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Valuation Risk Revalued

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Abstract

The recent asset pricing literature finds valuation risk is an important determinant of key asset pricing moments. Valuation risk is modelled as a time preference shock within Epstein-Zin recursive utility preferences. While this form of valuation risk appears to fit the data extremely well, we show the preference specification violates an economically meaningful restriction on the weights in the Epstein-Zin time-aggregator. The same model with the corrected preference specification performs nearly as well at matching asset pricing moments, but only if the risk aversion parameter is well above the accepted range of values used in the literature. When the corrected preference specification is combined with Bansal-Yaron long-run risk, the estimated model significantly downgrades the role of valuation risk in determining asset prices. The only significant contribution of valuation risk is to help match the volatility of the risk-free rate.

Keywords: Epstein-Zin Utility; Valuation Risk; Equity Premium Puzzle; Risk-Free Rate Puzzle
JEL Classifications: D81; G12

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1 INTRODUCTION

In standard asset pricing models, uncertainty enters through the supply side of the economy, either through endowment shocks in a Lucas tree model or productivity shocks in a production economy model. Beginning with Albuquerque et al. (2016)—henceforth AELR—the asset pricing literature introduced demand side uncertainty or “valuation risk” as a potential explanation of key asset pricing puzzles. In macroeconomic parlance, these features are often referred to as discount factor or time preference shocks.[1] AELR and other recent papers (e.g., Creal and Wu (2017); Schorfheide et al. (2018)) contend that valuation risk is an important determinant of key asset pricing moments.

We show the success of valuation risk in resolving various asset pricing puzzles rests sensitively on the way the preference shock enters the utility function. de Groot et al. (2018) show that within Epstein and Zin (1991) recursive utility preferences, the time-varying weights in the CES time-aggregator must sum to 1 to eliminate asymptotic dynamics in the model. The specification introduced by AELR and used in the subsequent literature fails this economically important restriction.[2]

This paper corrects the preferences used in this class of models and re-evaluates the role of valuation risk in resolving classic asset pricing puzzles. While the correction will appear minor, it profoundly changes the predictions of the model. Key comparative statics, such as the response of the equity premium and the risk-free rate to a rise in the intertemporal elasticity of substitution (IES) parameter, switch sign. This means that once we re-estimate the model, the parameters that best fit the data change dramatically. For example, our baseline model with the corrected preferences requires a coefficient of relative risk aversion (RA) well above the accepted range in the literature.

For intuition, consider the log-stochastic discount factor (SDF) under Epstein-Zin preferences

$$\hat{m}_{t+1} = \theta \log \beta + \theta \underbrace{(\omega \hat{a}_{t+1} - \hat{a}_t)}_{\text{valuation risk}} - (\theta/\psi)\Delta \hat{c}_{t+1} + (\theta - 1)\hat{r}_{y,t+1}, \tag{1}$$

where the first, third, and fourth terms—subjective discount factor (β), log-consumption growth ($\Delta \hat{c}_{t+1}$), and return on the endowment ($\hat{r}_{y,t+1}$)—are standard in this class of asset pricing models. The second term captures valuation risk, where $\hat{a}_t$ is a time preference shock. In AELR and the subsequent literature, $\omega = 1$. Once we correct the preferences and re-derive the log-SDF, we find $\omega = \beta$. Since β is the subjective discount factor at a monthly frequency, it is very close to 1. Therefore, at first sight, this innovation appears innocuous. However, when we apply this single, seemingly minor, alteration to the model, the asset pricing predictions are starkly different. In particular, it becomes difficult to resolve the equity premium (Mehra and Prescott (1985)), risk-free rate (Weil (1989)) and correlation puzzles (Campbell and Cochrane (1999)) with the corrected preferences.

[1] Discount factor shocks have become common in the business cycle literature since the 2007 financial crisis because they are an effective reduced-form mechanism for getting to the zero lower bound on the nominal interest rate.

[2] Rapach and Tan (2018) estimate a production asset pricing model with the specification in de Groot et al. (2018).

The problem with the original valuation risk specification is related to the preference parameter $\theta \equiv (1-\gamma)/(1-1/\psi)$ that enters the log-SDF, where γ is RA and ψ is the IES. Under constant relative risk aversion (CRRA) preferences, $\gamma = 1/\psi$. In this case, $\theta = 1$ and the log-SDF becomes

$$\hat{m}_{t+1} = \log\beta + (\omega\hat{a}_{t+1} - \hat{a}_t) - \Delta\hat{c}_{t+1}/\psi. \tag{2}$$

The return on the endowment drops out of (1), so the log-SDF is simply composed of the subjective discount factor and consumption growth terms. The benefit of Epstein-Zin preferences is that they decouple γ and ψ, so it is possible to simultaneously have high RA and a high IES. However, there is a highly nonlinear relationship between θ and ψ, as shown in figure 1. A vertical asymptote occurs at $\psi = 1$: θ tends to infinity as ψ approaches 1 from below while the opposite occurs as ψ approaches 1 from above. In fact, θ is undefined when the IES equals 1. In addition to the vertical asymptote, there is also a horizontal asymptote at $1-\gamma$ as the IES becomes perfectly elastic.

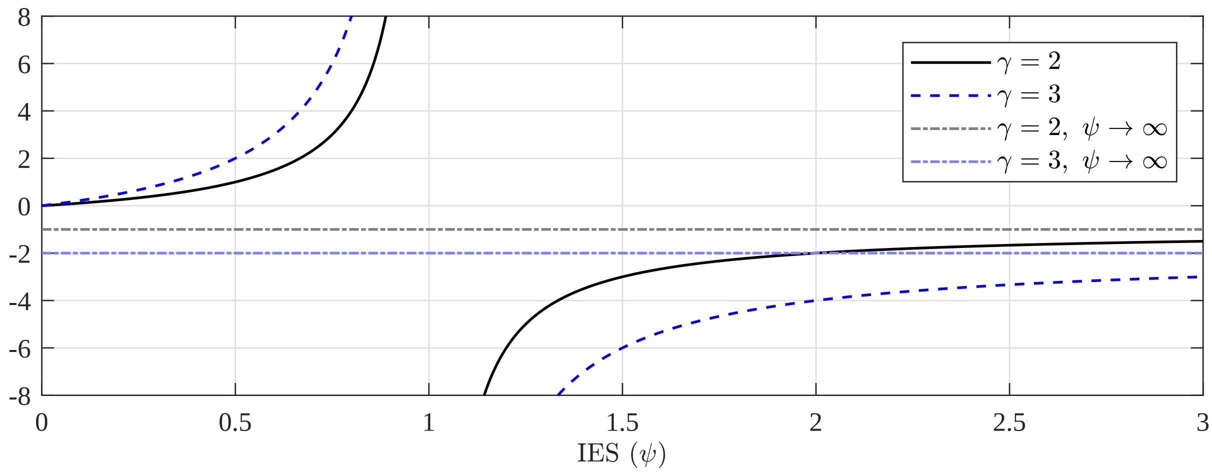

Figure 1: Preference parameter θ in the stochastic discount factor from a model with Epstein-Zin preferences.

Under the original Epstein and Zin (1989) preferences and the generalization in de Groot et al. (2018) to include time-varying valuation risk, the asymptote in figure 1 does not affect asset pricing behavior. Moreover, there is a well-defined equilibrium when the IES equals 1 and the asset pricing predictions are robust to small variations in the IES around 1. Continuity is preserved because the weights in the time-aggregator always sum to unity. An alternative interpretation of this result is that the time-aggregator maintains the well-known property that a CES aggregator tends to a Cobb-Douglas aggregator as the elasticity approaches 1. In the original AELR valuation risk specification, the restriction on the weights is violated so the limiting properties of the CES aggregator break down. As a consequence, the asymptote in figure 1 permeates asset pricing behavior, with small variations in the IES above and below unity generating very different asset pricing outcomes.

Interestingly, the spurious asymptote that occurs with the original valuation risk specification helps match key asset pricing moments. Furthermore, when we estimate a model that includes both valuation risk and long-run risk following Bansal and Yaron (2004), counterfactual exercises demonstrate that the asset pricing moments are almost completely explained by valuation risk, not long-run risk. The reason is straightforward. The asymptote stemming from θ allows the model to deliver an arbitrarily large equity premium and an arbitrarily low risk-free rate as the IES approaches 1 from above.[3] With the corrected preferences, valuation risk has a much smaller role.

We summarize our main results as follows: (1) The original valuation risk model is specified incorrectly but does well in matching asset pricing moments; (2) The corrected valuation risk model performs nearly as well at matching asset pricing moments, but only if the RA parameter is well above the typical range of values used in the literature; (3) When the corrected valuation risk specification is combined with long-run risk, the estimated model significantly downgrades the role of valuation risk in determining asset prices. For example, valuation risk alone generates an equity premium of 5.6% in the original model specification but only 1.8% in the corrected specification. (4) The main role of valuation risk in the correctly specified model is to generate sufficient volatility in the risk-free rate, a dimension along which long-run risk models generally perform poorly.

The paper proceeds as follows. Section 2 describes the baseline asset pricing model and the corrected valuation risk preference specification. Section 3 explains why asset prices depend so dramatically on the way valuation risk enters the Epstein-Zin utility function. Section 4 quantifies the effects of the corrected valuation risk specification on parameter estimates and asset prices. Section 5 estimates the relative importance of valuation and long-run risk. Section 6 concludes.

2 BASELINE ASSET-PRICING MODEL

We begin by laying out our baseline model. There are two assets: an endowment share, $s_{1,t}$, which pays income, y_t, and is in fixed unit supply, and an equity share, $s_{2,t}$, which pays dividends, d_t, and is in zero net supply. The representative agent chooses sequences $\{c_t, s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ to maximize

$$U_t^{AELR} = [a_t^{AELR}(1-\beta)c_t^{(1-\gamma)/\theta} + \beta(E_t[(U_{t+1}^{AELR})^{1-\gamma}])^{1/\theta}]^{\theta/(1-\gamma)}, \quad 1 \neq \psi > 0, \tag{3}$$

as proposed by Albuquerque et al. (2016), or

$$U_t^{DRT} = \begin{cases} [(1-a_t^{DRT}\beta)c_t^{(1-\gamma)/\theta} + a_t^{DRT}\beta(E_t[(U_{t+1}^{DRT})^{1-\gamma}])^{1/\theta}]^{\theta/(1-\gamma)}, & \text{for } 1 \neq \psi > 0, \\ c_t^{1-a_t^{DRT}\beta}(E_t[(U_{t+1}^{DRT})^{1-\gamma}])^{a_t^{DRT}\beta/(1-\gamma)}, & \text{for } \psi = 1, \end{cases} \tag{4}$$

as in de Groot et al. (2018)—henceforth DRT. The key difference between (3) and (4) is as follows:

> *The weights of the time-aggregator in (3), $a_t^{AELR}(1-\beta)$ and β, do not sum to 1, whereas the weights in (4), $(1-a_t^{DRT}\beta)$ and $a_t^{DRT}\beta$, do sum to 1.*

[3] The conceptual issue with the original valuation risk specification is that an IES marginally below one creates the opposite result—an arbitrarily large and *negative* equity premium with an arbitrarily large and positive risk-free rate.

The representative agent's choices are constrained by the flow budget constraint given by

$$c_t + p_{y,t}s_{1,t} + p_{d,t}s_{2,t} = (p_{y,t} + y_t)s_{1,t-1} + (p_{d,t} + d_t)s_{2,t-1}, \tag{5}$$

where $p_{y,t}$ and $p_{d,t}$ are the endowment and dividend claim prices. The optimality conditions imply

$$E_t[m^j_{t+1} r_{y,t+1}] = 1, \quad r_{y,t+1} \equiv (p_{y,t+1} + y_{t+1})/p_{y,t}, \tag{6}$$
$$E_t[m^j_{t+1} r_{d,t+1}] = 1, \quad r_{d,t+1} \equiv (p_{d,t+1} + d_{t+1})/p_{d,t}, \tag{7}$$

where $r_{y,t+1}$ and $r_{d,t+1}$ are the gross rates of return on the endowment and equity claim and

$$m^{AELR}_{t,t+1} \equiv \beta \left(\frac{a^{AELR}_{t+1}}{a^{AELR}_t}\right) \left(\frac{c_{t+1}}{c_t}\right)^{-1/\psi} \left(\frac{(V^{AELR}_{t+1})^{1-\gamma}}{E_t[(V^{AELR}_{t+1})^{1-\gamma}]}\right)^{1-\frac{1}{\theta}}, \tag{8}$$
$$m^{DRT}_{t,t+1} \equiv a^{DRT}_t \beta \left(\frac{1 - a^{DRT}_{t+1}\beta}{1 - a^{DRT}_t\beta}\right) \left(\frac{c_{t+1}}{c_t}\right)^{-1/\psi} \left(\frac{(V^{DRT}_{t+1})^{1-\gamma}}{E_t[(V^{DRT}_{t+1})^{1-\gamma}]}\right)^{1-\frac{1}{\theta}}. \tag{9}$$

To permit an approximate analytical solution, we rewrite (6) and (7) as follows

$$E_t[\exp(\hat{m}^j_{t+1} + \hat{r}_{y,t+1})] = 1, \tag{10}$$
$$E_t[\exp(\hat{m}^j_{t+1} + \hat{r}_{d,t+1})] = 1, \tag{11}$$

where $\hat{m}^j_{t+1}$ is defined in (1) and $\hat{a}_t \equiv \hat{a}^{AELR}_t = -\hat{a}^{DRT}_t/(1-\beta)$ so the shocks in the two models are directly comparable. The common time preference shock, a_{t+2}, evolves according to

$$\Delta\hat{a}_{t+2} = \rho_a \Delta\hat{a}_{t+1} + \sigma_a \varepsilon^a_{t+1}, \ \varepsilon^a_{t+1} \sim \mathbb{N}(0,1), \tag{12}$$

where $0 \leq \rho_a < 1$, $\sigma_a \geq 0$ is the shock standard deviation, a hat denotes the log of a variable, and Δ denotes a first-difference.[4] We then apply a Campbell and Shiller (1988) approximation to obtain

$$\hat{r}_{y,t+1} = \kappa_{y0} + \kappa_{y1}\hat{z}_{y,t+1} - \hat{z}_{y,t} + \Delta\hat{y}_{t+1}, \tag{13}$$
$$\hat{r}_{d,t+1} = \kappa_{d0} + \kappa_{d1}\hat{z}_{d,t+1} - \hat{z}_{d,t} + \Delta\hat{d}_{t+1}, \tag{14}$$

where $\hat{z}_{y,t+1}$ is the price-endowment ratio, $\hat{z}_{d,t+1}$ is the price-dividend ratio, and

$$\kappa_{y0} \equiv \log(1 + \exp(\hat{z}_y)) - \kappa_{y1}\hat{z}_y, \quad \kappa_{y1} \equiv \exp(\hat{z}_y)/(1 + \exp(\hat{z}_y)), \tag{15}$$
$$\kappa_{d0} \equiv \log(1 + \exp(\hat{z}_d)) - \kappa_{d1}\hat{z}_d, \quad \kappa_{d1} \equiv \exp(\hat{z}_d)/(1 + \exp(\hat{z}_d)), \tag{16}$$

are constants that are functions of the steady-state price-endowment and price-dividend ratio.

[4]The DRT preferences place a bound on a_t. Specifically, $0 < a_t < 1/\beta$. Given the process in (12), a_t will exceed the bound in finite time, since the variance of a_t is increasing in t. We decided to stick with (12) to follow the literature. Results with a stationary AR(2) process for a_t that respects the bound up to a tolerance are available upon on request.

To close the model, we assume the processes for endowment and dividend growth are given by

$$\Delta\hat{y}_{t+1} = \mu + \sigma_y \varepsilon^y_{t+1},\ \varepsilon^y_{t+1} \sim \mathbb{N}(0,1), \tag{17}$$

$$\Delta\hat{d}_{t+1} = \mu + \pi_{dy}\sigma_y \varepsilon^y_{t+1} + \psi_d\sigma_y\varepsilon^d_{t+1},\ \varepsilon^d_{t+1} \sim \mathbb{N}(0,1), \tag{18}$$

where μ is the common growth rate of the two assets, $\sigma_y \geq 0$ and $\psi_d\sigma_y \geq 0$ are the shock standard deviations, and π_{dy} captures the correlation between consumption growth and dividend growth.[5] Asset market clearing implies $s_{1,t} = 1$ and $s_{2,t} = 0$, so the resource constraint is given by $\hat{c}_t = \hat{y}_t$.

Equilibrium consists of sequences of quantities $\{\hat{c}_t\}_{t=0}^{\infty}$, prices $\{\hat{m}_{t+1}, \hat{z}_y, \hat{z}_d, \hat{r}_{y,t+1}, \hat{r}_{d,t+1}\}_{t=0}^{\infty}$ and exogenous variables $\{\hat{y}_t, \hat{d}_t, \hat{a}_t\}_{t=0}^{\infty}$ that satisfy (1), (10)-(14), (17), (18), and the resource constraint, given the state of the economy, $\{\hat{a}_{t+1}, \hat{a}_t\}$, and sequences of shocks, $\{\varepsilon_{y,t}, \varepsilon_{d,t}, \varepsilon_{a,t}\}_{t=1}^{\infty}$.

We posit the following solutions for the price-endowment and price-dividend ratios:

$$\hat{z}_{y,t} = \eta_{y0} + \eta_{y1}\hat{a}_{t+1} + \eta_{y2}\hat{a}_t, \tag{19}$$

$$\hat{z}_{d,t} = \eta_{d0} + \eta_{d1}\hat{a}_{t+1} + \eta_{d2}\hat{a}_t, \tag{20}$$

where $\hat{z}_y = \eta_{y0}$ and $\hat{z}_d = \eta_{d0}$. We solve the model using the method of undetermined coefficients. Appendix A derives the equilibrium conditions, the solution, and closed-form asset-prices.

3 INTUITION

This section develops intuition for why the valuation risk specification has such large effects on the model predictions. To simplify the exposition, we consider different stylized shock processes.

3.1 CONVENTIONAL MODEL First, it is useful to review the role of Epstein-Zin preferences and the separation of the RA and IES parameters in matching the risk-free rate and equity premium. For simplicity, we remove valuation risk ($\sigma_a = 0$) and assume endowment/dividend risk is perfectly correlated ($\psi_d = 0$; $\pi_{dy} = 1$). The average risk-free rate and average equity premium are given by

$$E[r_f] = -\log\beta + \mu/\psi + ((1/\psi - \gamma)(1-\gamma) - \gamma^2)\sigma_y^2/2, \tag{21}$$

$$E[ep] = (2\gamma - 1)\sigma_y^2/2, \tag{22}$$

where the first term in (21) is the subjective discount factor, the second term accounts for endowment growth, and the third term accounts for precautionary savings. Endowment growth ($\mu > 0$) creates an incentive for agents to borrow in order to smooth consumption. Since bonds are in zero net supply, the risk-free rate must rise to deter borrowing. When the IES, ψ, is high, agents are willing to accept higher consumption growth so the compensation required to dissuade borrowing

[5] We use this specification to illustrate the role of valuation risk. In section 5, we add long-run risk to (17) and (18).

is lower. Therefore, the model requires a fairly high IES to match the low risk-free rate in the data.

With CRRA preferences, higher RA lowers the IES and pushes up the risk-free rate. With Epstein-Zin preferences, these parameters are independent, so a high IES can lower the risk-free rate without lowering RA. Notice the equity premium only depends on RA. Therefore, the model generates a low risk-free rate and modest equity premium with sufficiently high RA and IES parameter values. Of course, there is an upper bound on what constitute reasonable RA and IES values, which is the source of the risk-free rate and equity premium puzzles. Other prominent features such as long-run risk and stochastic volatility à la Bansal and Yaron (2004) help resolve these puzzles.

3.2 ORIGINAL VALUATION RISK MODEL

Now consider an example where we remove cash-flow risk ($\sigma_y = 0$) but keep valuation risk. For simplicity, we assume the time preference shock follows a random walk ($\rho_a = 0$). Under these assumptions, the return on the endowment and dividend claims are identical so $(\kappa_{y0}, \kappa_{y1}, \eta_{y0}, \eta_{y1}, \eta_{y2}) = (\kappa_{d0}, \kappa_{d1}, \eta_{d0}, \eta_{d1}, \eta_{d2}) \equiv (\kappa_0, \kappa_1, \eta_0, \eta_1, \eta_2)$. We first solve the model with the original AELR preferences, so the log-SDF is given by (1) with $\omega = 1$. With this specification, the average risk-free rate and average equity premium are given by

$$E[r_f] = -\log\beta + \mu/\psi + (\theta - 1)\kappa_1^2\sigma_a^2/2, \tag{23}$$

$$E[ep] = (1 - 2\theta)\kappa_1^2\sigma_a^2/2. \tag{24}$$

In this model, it is also straightforward to show the log-price-dividend ratio is given by $\hat{z}_t = \hat{z} + \hat{a}_{t+1} - \hat{a}_t$ (i.e., the loadings on $\hat{a}_{t+1}$ and $\hat{a}_t$ are 1 and -1). Therefore, when the agent becomes more patient and $\hat{a}_{t+1}$ rises, the price-dividend ratio jumps one-for-one and then returns to the stationary equilibrium in the next period. Since η_1 is independent of the IES, there is no endogenous mechanism that prevents the asymptote in θ from influencing the risk-free rate or equity premium. It is easy to see from (16) that $0 < \kappa_1 < 1$. Therefore, θ dominates the risk-free rate and equity return when the IES is near 1. The following result describes the comparative statics with the IES:

> *As ψ tends to 1 from above, θ tends to $-\infty$. As a result, the average risk-free rate tends to $-\infty$ and the average equity premium tends to $+\infty$.*

This key finding illustrates why valuation risk seems like such an attractive feature for jointly resolving the risk-free rate and equity premium puzzles. As the IES tends to 1 from above, θ becomes increasingly negative, which dominates other determinants of the risk-free rate and equity premium. In particular, with an IES slightly above 1, the asymptote in θ causes the average risk-free rate to become arbitrarily small, while making the average equity premium arbitrarily large. Bizarrely, an IES marginally below 1 (a popular value in the macro literature), generates the exact opposite predictions. Even when the IES is above 1 and away from the vertical asymptote, figure 1 shows θ can have a meaningful effect on asset prices given a large enough risk aversion parameter.

An IES equal to 1 is a key value in the asset pricing literature. For example, it is the basis of the "risk-sensitive" preferences in Hansen and Sargent (2008, section 14.3). Therefore, it is clearly a desirable property for small perturbations around an IES of 1 to not materially alter the predictions of the model. A well-known example of where this property holds is the standard Epstein-Zin asset pricing model without valuation risk. Even though the log-SDF as written in (2) is undefined when the IES equals 1, both the risk-free rate and the equity premium in (21) and (22) are well-defined.

3.3 CORRECTED VALUATION RISK MODEL When we correct the preferences, so the weights in the time-aggregator sum to 1, the average risk-free rate and equity risk premium are given by

$$E[r_f] = -\log\beta + \mu/\psi + (\theta - 1)(\kappa_1\eta_1)^2\sigma_a^2/2, \tag{25}$$

$$E[ep] = (1 - 2\theta)(\kappa_1\eta_1)^2\sigma_a^2/2, \tag{26}$$

which are the same as the original valuation risk model, except the loading η_1 appears. This parameter determines the response of the price-dividend ratio to an $\hat{a}_{t+1}$ shock, and is no longer invariant to the IES. In particular, $\hat{z}_t = \hat{z} + \eta_1\hat{a}_{t+1} - \hat{a}_t$. It is clear from (25) and (26) that for the asymptote to disappear, η_1 must equal 0 when the IES equals 1. Appendix A verifies this is true.

Why does the influence of the asymptote disappear when the IES equals 1? The response of the price-dividend ratio to an anticipated change in $\hat{a}_{t+1}$ is determined by the relative strength of the substitution and wealth effects. First, consider the substitution effect. A higher $\hat{a}_{t+1}$ means the agent values present consumption more relative to the future and therefore wants to consume more today by reducing saving.[6] This effect lowers current asset demand and the price-dividend ratio.

The wealth effect operates in the opposite direction. When $\hat{a}_{t+1}$ is higher, the rise in the agent's value of c_t is less than the fall in the value of future certainty equivalent consumption since consumption is expected to grow. Therefore, the agent feels poorer, causing current asset demand and the price-dividend ratio to rise. When the IES equals 1, the substitution and wealth effects cancel out. This means the price-dividend ratio and the *ex-post* return on equity does not react on impact to an anticipated change in $\hat{a}_{t+1}$, which eliminates the effects of the asymptote. When the IES exceeds 1, as is typically the case in asset pricing models, the substitution effect dominates and reduces current asset demand on impact, causing the price-dividend ratio to fall. In the special case when there is no consumption growth, there are no wealth effects of time preference shocks, and substitution effects do not occur until the change in $\hat{a}_{t+1}$ materializes and lowers the discount rate.

3.4 GRAPHICAL ILLUSTRATION Our analytical results show the way a time preference shock enters Epstein-Zin recursive utility determines whether the asymptote in θ shows up in equilibrium outcomes. Figure 2 illustrates our results by plotting the average risk-free rate, the average equity

[6] With the corrected preferences, a rise in $\hat{a}_{t+1}$ corresponds to a fall in a_{t+1}^{DRT}, so the agent becomes more impatient.

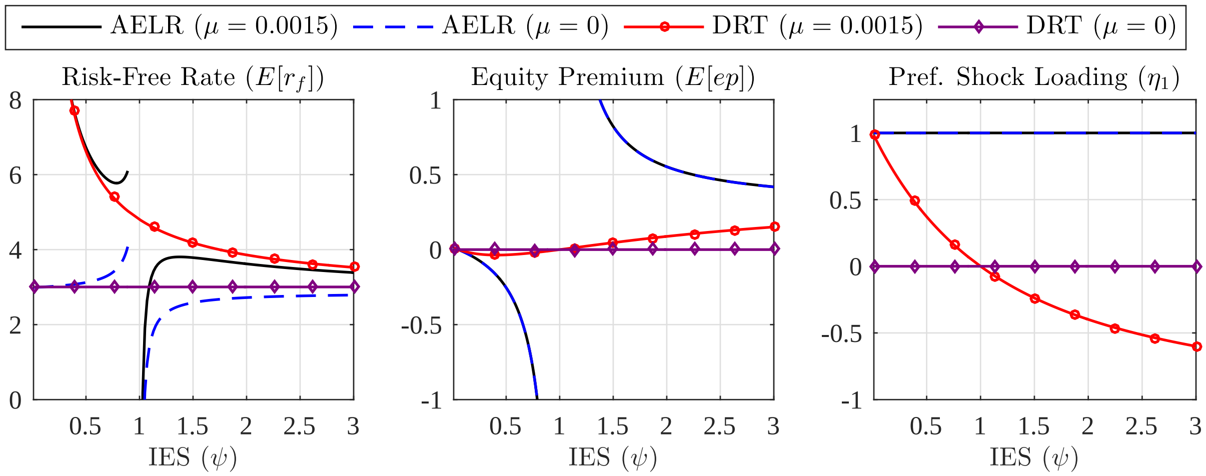

Figure 2: Equilibrium outcomes in the model without cashflow risk under AELR and DRT preferences.

premium, and the price-dividend ratio loading on the preference shock as a function of ψ. We focus on the example without cashflow risk ($\sigma_y = 0$). We plot the results under both preferences with and without growth. For illustrative purposes, we set $\beta = 0.9975$, $\gamma = 10$, and $\sigma_a = 0.005$.

With the AELR preferences, the average risk-free rate and average equity premium exhibit a vertical asymptote when the IES equals 1, regardless of whether μ is positive. As a result, the risk-free rate approaches positive infinity as the IES approaches 1 from below and negative infinity as the IES approaches 1 from above. The equity premium has the same comparative statics, except with the opposite sign. These results occur because $\eta_1 = 1$, regardless of the value of the IES. Therefore, the volatility of the return on equity is independent of the IES, while the volatility of the stochastic discount factor becomes infinitely large. This means the paradoxical agent with these preferences will sacrifice an infinite amount of consumption in order to hold an asset with zero risk.

In contrast, with DRT preferences the risk-free rate and equity premium are continuous in the IES, regardless of the value of μ. When $\mu = 0$, the endowment stream is constant. This means there is no incentive to smooth consumption, the average risk-free rate is independent of the IES, and there is no immediate response of the price-dividend ratio to a time-preference shock. As a consequence, there are no unanticipated changes in the equity return, and the average equity premium is zero. When $\mu > 0$, the agent has an incentive to smooth consumption, so the SDF and the return on the equity become correlated. When $\psi > 1$, the substitution effect from the preference shock dominates the wealth effect. This causes the price-dividend ratio to fall ($\eta_1 < 0$) when the SDF falls and leads to a positive equity premium. In this case, the comparative static effect of the equity premium to a change in the IES has the opposite sign in the corrected valuation risk model compared to the original valuation risk model. In the corrected model, the equity premium is rising in the IES, whereas in the original model it is falling in the IES. This is because the asymptote dominates the determination of asset prices in the original model, even for large IES values. When $\psi < 1$, the wealth effect dominates the substitution effect, so the price-dividend ratio rises ($\eta_1 > 0$) when the SDF falls. This generates a negative valuation risk equity premium. Finally, when $\psi = 1$, the substitution and wealth effects cancel out, leaving the price-dividend ratio unchanged ($\eta_1 = 0$). As a result, valuation risk generates no equity premium when the IES is 1.

4 Estimated Baseline Model

This section returns to the baseline model in section 2, which has valuation risk and stochastic endowment and dividend growth. We estimate the model with both the AELR and DRT preference specifications and then show how the parameter estimates and key asset pricing moments differ.

4.1 Data and Estimation Method We follow the estimation method in Albuquerque et al. (2016) and use their dataset, which contains annual observations from 1929 to 2011 of U.S. per-capita real consumption, the real market log return, the risk-free rate, per-capita real dividends, and the log price-dividend ratio. We estimate the model in two stages. In the first stage, we use Generalized Method of Moments (GMM) to obtain point estimates and a covariance matrix of key moments in the data. In the second stage, we use Simulated Method of Moments (SMM) to search for a parameter vector that minimizes the distance between the GMM point estimates and median short-sample model moments, weighted by the GMM estimate of the variance-covariance matrix. We use simulated annealing to minimize the objective function, J, since gradient-based techniques did not sufficiently explore the parameter space. A smaller J value indicates a better fit to the data.

The algorithm matches the following moments: the mean and standard deviation of consumption growth, dividend growth, real stock returns, the real risk-free rate, and the price-dividend ratio, the autocorrelation of the price-dividend ratio and real risk-free rate, the correlation between dividend growth and consumption growth, the correlation between equity returns and both consumption and dividend growth at a 1-, 5-, and 10-year horizon. See Appendix B for more details.

4.2 Parameter Estimates and Moments Table 1a shows the estimated parameter values and table 1b reports the data and model moments under the original and corrected valuation risk specifications. The AELR estimates are similar to the values reported in Albuquerque et al. (2016).[7] The model fits the data extremely well, with a lower J value than the DRT model. The AELR model requires a remarkably low RA value (2.2) but has a fairly typical value for the IES (2.3). The low RA value is due to the asymptote in the AELR preference specification. An IES close to 1 reduces the risk-free rate and raises the equity premium to an arbitrarily large extent. Therefore, the AELR model is able to maintain an extremely low RA value and still match the data.

[7] Our results differ from AELR in two ways. One, AELR restrict their SMM procedure to exactly match the average risk-free rate. We do not apply that restriction and instead weight by the GMM variance-covariance matrix. Two, even with the restriction in place, our simulated annealing procedure was able to achieve a lower J value than AELR report.

Parameter	AELR	DRT	Parameter	AELR	DRT	Parameter	AELR	DRT
γ	2.19813	254.26306	σ_y	0.00786	0.00421	π_{dy}	−0.31831	0.39380
ψ	2.29054	9.31110	μ	0.00151	0.00219	σ_a	0.00068	0.00012
β	0.99740	0.99210	ψ_d	2.00940	4.46431	ρ_a	0.98806	0.99886

(a) Parameter estimates. AELR: $J = 5.15$; DRT: $J = 10.70$.

		AELR Specification			DRT Specification		
Moment	Data	All Shocks	Only CFR	Only VR	All Shocks	Only CFR	Only VR
$E[r_d]$	7.83	6.04	3.58	6.30	11.85	8.49	9.97
$SD[r_d]$	17.25	15.55	5.42	14.41	9.95	6.29	3.84
$E[r_f]$	0.13	0.24	3.82	0.34	0.69	6.81	2.68
$SD[r_f]$	3.56	4.56	0.00	4.56	4.49	0.00	4.49
$E[ep]$	7.70	5.80	−0.24	5.97	11.16	1.68	7.30
$SD[z_d]$	0.47	0.26	0.03	0.25	0.77	0.04	0.52
$AC[r_f]$	0.52	0.88	1.00	0.88	1.00	1.00	1.00

(b) Unconditional short-sample moments given the parameter estimates for each model. "All Shocks" simulates the model with all of the shocks, "Only CFR" simulates the model with only the cashflow risk shocks, and "Only VR" simulates the model with only the valuation risk shocks.

Table 1: Baseline model estimates

Corrected valuation risk behaves more like typical cash flow risk, in that both the risk-free rate and the equity premium are increasing in the IES. As a result, the corrected valuation risk IES (9.3) is higher than the original valuation risk IES (2.3). The higher IES diminishes the consumption smoothing motive and lowers the risk-free rate. However, since the higher IES is not able to sufficiently raise the covariance between the SDF and the equity return, the data requires much higher RA. Our RA estimate (254) is an order of magnitude larger than the upper bound usually accepted in the literature.[8] This causes the model to underpredict the variance of the equity return and overpredict the variance of the risk-free rate. In addition, to make the valuation risk shocks more important, the data prefers a higher average growth rate of dividends because it amplifies the effect of a time preference shock. The data also prefers highly persistent valuation risk shocks, which raise the equity premium because agents value an early resolution of uncertainty with $\gamma > 1/\psi$. However, the higher persistence also causes the model to overpredict the autocorrelation of the risk-free rate.

Next, we decompose the relative role of valuation risk and cash flow risk in explaining the various asset pricing moments. Table 1b reports the model moments corresponding to counterfactual simulations that either remove valuation risk ("Only CFR") or cashflow risk ("Only VR") from the model. In each case, we re-solve the model after setting $\sigma_a = \rho_a = 0$ for "Only CFR" and $\sigma_y = \psi_d = \pi_{dy} = 0$ for "Only VR", so that agents make decisions subject to only one type of risk.

[8] Mehra and Prescott (1985) suggest restricting RA to be a maximum of 10. The acceptable range for the IES is less clearly defined in the literature although values above 3 are atypical. We can achieve a similar J value (11.70) as table 1 with a much lower RA (34.87) and a higher IES (27.36). In this case, the SMM algorithm is prioritizing matching the risk-free rate (0.20) over other moments. Both sets of estimates are well outside norms in the literature.

With the AELR specification, cashflow risk by itself generates almost no equity premium or precautionary savings demand because the RA parameter is so low. Therefore, the average risk-free rate is much higher than in the data. Without serial correlation in cash flow growth, cash flow risk alone is unable to generate movements in the risk-free rate. As a result, it is valuation risk and the effects of the embedded asymptote that are able to match all of the asset pricing moments.

With the corrected valuation risk specification, the model still matches the asset pricing moments reasonably well as long as one accepts the high RA and IES values. Cashflow risk by itself fails to lower the risk-free rate. Therefore, the equity premium is significantly lower than with the AELR specification. With the high RA and IES parameters and the near unit root in the time preference shock process, valuation risk shocks lower the risk-free rate, explain more of the equity premium than cashflow risk shocks, and are the only source of volatility for the risk-free rate. In short, cashflow risk plays a bigger role in explaining asset pricing moments under the DRT specification, but the role of valuation risk is similarly important regardless of the preference specification.

5 ESTIMATED LONG-RUN RISK MODEL

In the baseline model, valuation risk explains most of the key asset pricing moments, even after correcting the preference specification. However, the prominent role of valuation risk is not much of a surprise given that we have abstracted from long-run risk, which is a well-known potential resolution of many asset pricing puzzles. Therefore, this section introduces long-run risk to our baseline model and re-examines the role of valuation risk with both preference specifications.

In order to introduce long-run risk, we modify (17) and (18) as follows

$$\Delta\hat{y}_{t+1} = \mu + \hat{x}_t + \sigma_y \varepsilon^y_{t+1},\ \varepsilon^y_{t+1} \sim \mathbb{N}(0,1), \tag{27}$$

$$\Delta\hat{d}_{t+1} = \mu + \phi_d \hat{x}_t + \pi_{dy}\sigma_y \varepsilon^y_{t+1} + \psi_d \sigma_y \varepsilon^d_{t+1},\ \varepsilon^d_{t+1} \sim \mathbb{N}(0,1), \tag{28}$$

$$\hat{x}_{t+1} = \rho_x \hat{x}_t + \psi_x \sigma_y \varepsilon_{x,t+1},\ \varepsilon^x_{t+1} \sim \mathbb{N}(0,1), \tag{29}$$

where the specification of the persistent component, $\hat{x}_t$, which is common to both the endowment and dividends growth processes, follows Bansal and Yaron (2004). We apply the same estimation procedure as the baseline model, except we estimate three additional parameters, ϕ_d, ρ_x, and ψ_x.[9]

Table 2a shows the estimated parameters and table 2b reports key asset pricing moments for the model with long-run risk. In the AELR model, the presence of long-run risk provides a slightly better fit of the data (the J value declines from 5.15 to 4.31). Both the RA (1.6) and IES (1.4) parameter values are lower than in the baseline model. However, long-run risk plays a minor role since the asymptote resulting from the valuation risk specification continues to dominate the deter-

[9]Long-run risk adds one additional state variable, $\hat{x}_t$. Following the guess and verify procedure applied to the baseline model, we use Mathematica to solve for unknown coefficients in the price-endowment and price-dividend ratios.

Parameter	AELR	DRT	Parameter	AELR	DRT	Parameter	AELR	DRT
γ	1.63436	15.04390	μ	0.00164	0.00157	ρ_a	0.99180	0.94072
ψ	1.43058	1.88177	ψ_d	1.84323	1.73255	ϕ_d	2.73398	7.12070
β	0.99761	0.99918	π_{dy}	−0.93561	−0.25092	ρ_x	0.95732	0.99354
σ_y	0.00615	0.00628	σ_a	0.00049	0.00114	ψ_x	0.04933	0.01118

(a) Parameter estimates. AELR: $J = 4.31$; DRT: $J = 1.97$.

		AELR Specification			DRT Specification		
Moment	Data	All Shocks	Only CFR	Only VR	All Shocks	Only CFR	Only VR
$E[r_d]$	7.83	6.31	4.04	6.43	6.22	7.39	3.14
$SD[r_d]$	17.25	15.54	6.90	13.91	17.36	15.06	4.98
$E[r_f]$	0.13	0.89	4.21	0.97	0.74	1.05	1.21
$SD[r_f]$	3.56	3.90	0.79	3.79	3.52	0.32	3.50
$E[ep]$	7.70	5.42	−0.18	5.47	5.49	6.34	1.93
$SD[z_d]$	0.47	0.28	0.06	0.27	0.37	0.33	0.26
$AC[r_f]$	0.52	0.90	0.70	0.91	0.61	0.92	0.61

(b) Unconditional short-sample moments given the parameter estimates for each model. “All Shocks” simulates the model with all of the shocks, “Only CFR” simulates the model with only the cashflow risk shocks, and “Only VR” simulates the model with only the valuation risk shocks.

Table 2: Long-run risk model estimates

mination of asset prices. Valuation risk by itself explains almost all of the asset pricing moments, including the near-zero risk free rate and 5% equity premium. Without valuation risk, the model generates no equity premium, a risk-free rate near 4%, and standard deviations well below the data.

The results change dramatically in the corrected valuation risk model with long-run risk. In particular, there are three interesting results with DRT preferences. One, the model with long-run risk provides a substantially better fit of the data over the baseline model, as the J value falls from 10.70 to 1.97. Two, both the RA and IES parameter values are much lower than the values in the baseline model. For example, the RA parameter declines from 254 in the baseline model to 15 in the model with long-run risk, close to the acceptable range in the asset pricing literature. Three, valuation risk no longer explains the vast majority of asset pricing moments. In contrast with the AELR model, cashflow risk by itself generates an equity premium similar to the data. Valuation risk alone only generates a 1.9% equity premium. Interestingly, however, valuation risk still plays an important role because it explains the volatility of the risk-free rate. The standard deviation of the risk-free rate in the data is 3.6%. However, long-run risk alone only generates a standard deviation of 0.3%. In short, there is still a role for valuation risk, but in its corrected form, its role in resolving key asset pricing puzzles is much smaller in the presence of long-run cashflow risk.

The Correlation Puzzle Another important asset pricing puzzle pertains to the correlation between equity returns and fundamentals (Cochrane and Hansen (1992)). In the data, the correlation between equity returns and consumption growth is near zero, regardless of the horizon. The correlation between equity returns and dividend growth is small over short horizons but increases over longer horizons. The central issue is that many asset-pricing models predict too strong of a correlation between stock returns and fundamentals relative to the data. Clearly, if valuation risk generates meaningful volatility in asset returns and yet is uncorrelated with consumption and dividend growth (as in the model above), then valuation risk has the potential to resolve the correlation puzzle.

		AELR Specification		DRT Specification	
Moment	Data	All Shocks	Only CFR	All Shocks	Only CFR
1-year $Corr[\Delta c, r_d]$	-0.07	-0.02	-0.03	0.02	0.02
5-year $Corr[\Delta c, r_d]$	-0.01	0.09	0.21	0.10	0.11
10-year $Corr[\Delta c, r_d]$	-0.08	0.17	0.35	0.21	0.23
1-year $Corr[\Delta d, r_d]$	0.08	0.34	0.78	0.28	0.31
5-year $Corr[\Delta d, r_d]$	0.22	0.36	0.79	0.28	0.31
10-year $Corr[\Delta d, r_d]$	0.51	0.45	0.90	0.41	0.44

Table 3: Unconditional short-sample moments given the parameter estimates for each model. "All Shocks" simulates the model with all of the shocks and "Only CFR" simulates the model with only the cashflow risk shocks.

Table 3 shows the correlations between equity returns and fundamentals over 1-, 5-, and 10-year horizons in the data and predicted by the model. We also consider a counterfactual with only cash-flow risk ("Only CFR"). The original AELR specification predicts near-zero correlations with consumption growth over short-horizons, but they increase over longer horizons. The correlation with dividend growth is stronger than the data over a 1-year horizon but increases and is closer to the data over a 10-year horizon. When we remove valuation risk, those same correlations are close to one.

The correlations between equity returns and both consumption and dividend growth are similar across the original and corrected valuation risk specifications. However, all of the correlations with the corrected specification are driven entirely by cash flow risk, rather than valuation risk. The intuition for this result is reflective of the results in table 2. In the model with long-run risk, most of the volatility in equity returns comes from changes in consumption and dividend growth, while valuation risk plays a secondary role. Therefore, valuation risk no longer reduces the correlations.

6 CONCLUSION

The way valuation risk enters Epstein-Zin recursive utility preferences has important implications for how a standard asset pricing model explains key asset pricing moments. Under the original AELR preferences, an asymptote in the parameter space with respect to the IES dominates equilibrium outcomes. In particular, the presence of the asymptote allows valuation risk to explain the historically low risk-free rate and high equity premium, but the theoretical foundations of the preference specification are suspect. Once we correct the preferences to remove the influence of the asymptote, valuation risk alone requires implausibly high risk aversion to match the data. When we add long-run risk to the model, we find a relatively small role for valuation risk in resolving asset pricing puzzles, in contrast with the findings in the literature. Corrected valuation risk however still plays an important role in generating volatility in the risk-free rate that is in line with the data.

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A Pricing Kernel Derivation and Model Solution

The value function for preference specification $j \in \{AELR, DRT\}$ is given by

$$\begin{aligned} V_t^j &= \max[w_{1,t}^j c_t^{(1-\gamma)/\theta} + w_{2,t}^j (E_t[(V_{t+1}^j)^{1-\gamma}])^{1/\theta}]^{\theta/(1-\gamma)} \\ &\quad - \lambda_t(c_t + p_{d,t}s_{1,t} + p_{y,t}s_{2,t} - (p_{d,t} + d_t)s_{1,t-1} - (p_{y,t} + y_t)s_{2,t-1}), \end{aligned}$$

where $w_{1,t}^{AELR} = a_t^{AELR}(1-\beta)$, $w_{1,t}^{DRT} = 1 - a_t^{DRT}\beta$, $w_{2,t}^{AELR} = \beta$, and $w_{2,t}^{DRT} = a_t^{DRT}\beta$.

The representative agent's optimality conditions imply

$$w_{1,t}^j (V_t^j)^{1/\psi} c_t^{-1/\psi} = \lambda_t, \tag{30}$$

$$w_{2,t}^j (V_t^j)^{1/\psi} (E_t[(V_{t+1}^j)^{1-\gamma}])^{1/\theta - 1} E_t[(V_{t+1}^j)^{-\gamma}(\partial V_{t+1}^j/\partial s_{1,t})] = \lambda_t p_{y,t}, \tag{31}$$

$$w_{2,t}^j (V_t^j)^{1/\psi} (E_t[(V_{t+1}^j)^{1-\gamma}])^{1/\theta - 1} E_t[(V_{t+1}^j)^{-\gamma}(\partial V_{t+1}^j/\partial s_{2,t})] = \lambda_t p_{d,t}, \tag{32}$$

where $\partial V_t^j/\partial s_{1,t-1} = \lambda_t(p_{y,t} + y_t)$ and $\partial V_t^j/\partial s_{2,t-1} = \lambda_t(p_{d,t} + d_t)$ by the envelope theorem. Updating the envelope conditions and combining (30)-(32) yields (8) and (9) in the main text.

Following Epstein and Zin (1991), we posit a minimum state variable solution of the form

$$V_t^j = \xi_{1,t}s_{1,t-1} + \xi_{2,t}s_{2,t-1}, \tag{33}$$

$$c_t = \xi_{3,t}s_{1,t-1} + \xi_{4,t}s_{2,t-1}. \tag{34}$$

where ξ is a vector of unknown coefficients. The envelope conditions combined with (30) imply

$$\xi_{1,t} = w_{1,t}^j (V_t^j)^{1/\psi} c_t^{-1/\psi}(p_{y,t} + y_t), \tag{35}$$

$$\xi_{2,t} = w_{1,t}^j (V_t^j)^{1/\psi} c_t^{-1/\psi}(p_{d,t} + d_t). \tag{36}$$

Multiplying the respective conditions by $s_{1,t-1}$ and $s_{2,t-1}$ and then adding yields

$$V_t^j = w_{1,t}^j (V_t^j)^{1/\psi} c_t^{-1/\psi}((p_{y,t} + y_t)s_{1,t-1} + (p_{d,t} + d_t)s_{2,t-1}), \tag{37}$$

which after plugging in the budget constraint, (5), can be written as

$$(V_t^j)^{(1-\gamma)/\theta} = w_{1,t}^j c_t^{-1/\psi}(c_t + p_{y,t}s_{1,t} + p_{d,t}s_{2,t}) = w_{1,t}^j c_t^{-1/\psi}(c_t + p_{y,t}). \tag{38}$$

Therefore, the optimal value function can be written as

$$w_{1,t}^j c_t^{-1/\psi} p_{y,t} = w_{2,t}^j (E_t[(V_{t+1}^j)^{1-\gamma}])^{1/\theta}. \tag{39}$$

Solving (38) for V_t^j and (39) for $E_t[(V_{t+1}^j)^{1-\gamma}]$ and then plugging into (8) and (9) implies

$$m_{t+1} = \beta(x_t^j)^{\theta}(c_{t+1}/c_t)^{-\theta/\psi} r_{y,t+1}^{\theta-1}, \tag{40}$$

where

$$x_t^{AELR} \equiv a_{t+1}^{AELR}/a_t^{AELR},$$
$$x_t^{DRT} \equiv a_t^{DRT}\beta(1-a_{t+1}^{DRT}\beta)/(1-a_t^{DRT}\beta).$$

Taking logs of (40) yields (1) in the main text, where

$$\hat{x}_t^{AELR} = \hat{a}_{t+1}^{AELR} - \hat{a}_t^{AELR},$$
$$\hat{x}_t^{DRT} = \hat{a}_t^{DRT} + \log(1-\beta\exp(\hat{a}_{t+1}^{DRT})) - \log(1-\beta\exp(\hat{a}_t^{DRT})) \approx -(\beta\hat{a}_{t+1}^{DRT} - \hat{a}_t^{DRT})/(1-\beta).$$

We define $\hat{a}_t \equiv \hat{a}_t^{AELR} = -\hat{a}_t^{DRT}/(1-\beta)$, so the preference shocks in the two models are directly comparable. It follows that $\hat{x}_t^j = \omega^j \hat{a}_{t+1} - \hat{a}_t$ just like in (1), where $\omega^{AELR} = 1$ and $\omega^{DRT} = \beta$.

The Campbell and Shiller (1988) approximation to the return on dividends is given by

$$\begin{aligned}
\hat{r}_{y,t+1} &= \log(y_{t+1}(p_{y,t+1}/y_{t+1}) + y_{t+1}) - \log(y_t(p_{y,t}/y_t)) \\
&= \log(y_{t+1}(\exp(\hat{z}_{y,t+1}) + 1)) - \hat{z}_{y,t} - \log(y_t) \\
&= \log(\exp(\hat{z}_{y,t+1}) + 1) - \hat{z}_{y,t} + \Delta\hat{y}_{t+1} \\
&\approx \log(\exp(\hat{z}_y) + 1) + \exp(\hat{z}_y)(\hat{z}_{y,t+1} - \hat{z}_y)/(1+\exp(\hat{z}_y)) - \hat{z}_{y,t} + \Delta\hat{y}_{t+1} \\
&= \kappa_{y0} + \kappa_{y1}\hat{z}_{y,t+1} - \hat{z}_{y,t} + \Delta\hat{y}_{t+1}.
\end{aligned}$$

The derivation for the return on the dividend, $\hat{r}_{d,t+1}$, is analogous.

We solve the model using a guess and verify method. For the endowment claim, we obtain

$$\begin{aligned}
0 &= \log(E_t[\exp(\hat{m}_{t+1} + \hat{r}_{y,t+1})]) \\
&= \log(E_t[\exp(\theta\hat{\beta} + \theta(\omega^j\hat{a}_{t+1} - \hat{a}_t) - (\theta/\psi)\Delta\hat{y}_{t+1} + \theta\hat{r}_{y,t+1})]) \\
&= \log(E_t[\exp(\theta\hat{\beta} + \theta(\omega^j\hat{a}_{t+1} - \hat{a}_t) + \theta(1-1/\psi)\Delta\hat{y}_{t+1} + \theta(\kappa_{y0} + \kappa_{y1}\hat{z}_{y,t+1} - \hat{z}_{y,t}))]) \\
&= \log\left(E_t\left[\exp\left(\begin{array}{c} \theta\hat{\beta} + \theta(\omega^j\hat{a}_{t+1} - \hat{a}_t) + \theta(1-1/\psi)(\mu + \sigma_y\varepsilon_{y,t+1}) + \theta\kappa_{y0} \\ +\theta\kappa_{y1}(\eta_{y0} + \eta_{y1}\hat{a}_{t+2} + \eta_{y2}\hat{a}_{t+1}) - \theta(\eta_{y0} + \eta_{y1}\hat{a}_{t+1} + \eta_{y2}\hat{a}_t) \end{array}\right)\right]\right) \\
&= \log\left(E_t\left[\exp\left(\begin{array}{c} \theta\hat{\beta} + \theta(1-1/\psi)\mu + \theta(\kappa_{y0} + \eta_{y0}(\kappa_{y1} - 1)) \\ +\theta(\omega^j + \eta_{y1}(\kappa_{y1}\tilde{\rho} - 1) + \eta_{y2}\kappa_{y1})\hat{a}_{t+1} \\ -\theta(1 + \eta_{y2} + \kappa_{y1}\eta_{y1}\rho_a)\hat{a}_t \\ +\theta(1-1/\psi)\sigma_y\varepsilon_{y,t+1} + \theta\kappa_{y1}\eta_{y1}\sigma_a\varepsilon_{a,t+1} \end{array}\right)\right]\right) \\
&= \theta\hat{\beta} + \theta(1-1/\psi)\mu + \theta(\kappa_{y0} + \eta_{y0}(\kappa_{y1} - 1)) + \tfrac{\theta^2}{2}((1-1/\psi)^2\sigma_y^2 + \kappa_{y1}^2\eta_{y1}^2\sigma_a^2) \\
&\quad + \theta(\omega^j + \eta_{y1}(\kappa_{y1}\tilde{\rho} - 1) + \eta_{y2}\kappa_{y1})\hat{a}_{t+1} - \theta(1 + \eta_{y2} + \kappa_{y1}\eta_{y1}\rho_a)\hat{a}_t,
\end{aligned}$$

where $\tilde{\rho} = 1 + \rho_a$. The last equality follows from the log-normality of $\exp(\varepsilon_y)$ and $\exp(\varepsilon_a)$.

After equating coefficients, we obtain the following exclusion restrictions

$$\hat{\beta} + (1 - 1/\psi)\mu + (\kappa_{y0} + \eta_{y0}(\kappa_{y1} - 1)) + \tfrac{\theta}{2}((1 - 1/\psi)^2\sigma_y^2 + \kappa_{y1}^2\eta_{y1}^2\sigma_a^2) = 0, \tag{41}$$

$$\omega^j + \eta_{y1}(\kappa_{y1}\tilde{\rho} - 1) + \eta_{y2}\kappa_{y1} = 0, \tag{42}$$

$$1 + \eta_{y2} + \kappa_{y1}\eta_{y1}\rho_a = 0. \tag{43}$$

For the dividend claim, we obtain

$$\begin{aligned}
0 &= \log(E_t[\exp(\hat{m}_{t+1} + \hat{r}_{d,t+1})]) \\
&= \log(E_t[\exp(\theta\hat{\beta} + \theta(\omega^j\hat{a}_{t+1} - \hat{a}_t) - (\theta/\psi)\Delta\hat{y}_{t+1} + (\theta - 1)\hat{r}_{y,t+1} + \hat{r}_{d,t+1})]) \\
&= \log\left(E_t\left[\exp\left(\begin{array}{c} \theta\hat{\beta} + \theta(\omega^j\hat{a}_{t+1} - \hat{a}_t) + (\theta(1 - 1/\psi) - 1)\Delta\hat{y}_{t+1} + \Delta\hat{d}_{t+1} \\ +(\theta - 1)(\kappa_{y0} + \kappa_{y1}\hat{z}_{y,t+1} - \hat{z}_{y,t}) + (\kappa_{d0} + \kappa_{d1}\hat{z}_{d,t+1} - \hat{z}_{d,t}) \end{array}\right)\right]\right) \\
&= \log\left(E_t\left[\exp\left(\begin{array}{c} \theta\hat{\beta} + \theta(\omega^j\hat{a}_{t+1} - \hat{a}_t) + (\theta(1 - 1/\psi) - 1)\Delta\hat{y}_{t+1} + \Delta\hat{d}_{t+1} \\ +(\theta - 1)(\kappa_{y0} + \kappa_{y1}(\eta_{y0} + \eta_{y1}\hat{a}_{t+2} + \eta_{y2}\hat{a}_{t+1}) - (\eta_{y0} + \eta_{y1}\hat{a}_{t+1} + \eta_{y2}\hat{a}_t)) \\ +\kappa_{d0} + \kappa_{d1}(\eta_{d0} + \eta_{d1}\hat{a}_{t+2} + \eta_{d2}\hat{a}_{t+1}) - (\eta_{d0} + \eta_{d1}\hat{a}_{t+1} + \eta_{d2}\hat{a}_t) \end{array}\right)\right]\right) \\
&= \log\left(E_t\left[\exp\left(\begin{array}{c} \theta\hat{\beta} + \theta(1 - 1/\psi)\mu + (\theta - 1)(\kappa_{y0} + \eta_{y0}(\kappa_{y1} - 1)) + (\kappa_{d0} + \eta_{d0}(\kappa_{d1} - 1)) \\ +(\theta\omega^j + (\theta - 1)[(\tilde{\rho}\kappa_{y1} - 1)\eta_{y1} + \kappa_{y1}\eta_{y2}] + (\tilde{\rho}\kappa_{d1} - 1)\eta_{d1} + \kappa_{d1}\eta_{d2})\hat{a}_{t+1} \\ -(\theta + (\theta - 1)\eta_{y2} + \eta_{d2} + ((\theta - 1)\kappa_{y1}\eta_{y1} + \kappa_{d1}\eta_{d1})\rho_a)\hat{a}_t \\ (\pi_{dy} - \gamma)\sigma_y\varepsilon_{y,t+1} + ((\theta - 1)\kappa_{y1}\eta_{y1} + \kappa_{d1}\eta_{d1})\sigma_a\varepsilon_{a,t+1} + \psi_d\sigma_y\varepsilon_{t+1}^d \end{array}\right)\right]\right) \\
&= \theta\hat{\beta} + \theta(1 - 1/\psi)\mu + (\theta - 1)(\kappa_{y0} + \eta_{y0}(\kappa_{y1} - 1)) + (\kappa_{d0} + \eta_{d0}(\kappa_{d1} - 1)) \\
&\quad + (\theta\omega^j + (\theta - 1)[(\tilde{\rho}\kappa_{y1} - 1)\eta_{y1} + \kappa_{y1}\eta_{y2}] + (\tilde{\rho}\kappa_{d1} - 1)\eta_{d1} + \kappa_{d1}\eta_{d2})\hat{a}_{t+1} \\
&\quad - (\theta + (\theta - 1)\eta_{y2} + \eta_{d2} + ((\theta - 1)\kappa_{y1}\eta_{y1} + \kappa_{d1}\eta_{d1})\rho_a)\hat{a}_t \\
&\quad + \tfrac{1}{2}((\pi_{dy} - \gamma)^2\sigma_y^2 + ((\theta - 1)\kappa_{y1}\eta_{y1} + \kappa_{d1}\eta_{d1})^2\sigma_a^2 + \psi_d^2\sigma_y^2),
\end{aligned}$$

Once again, equating coefficients implies the following exclusion restrictions

$$\begin{aligned}\theta\hat{\beta} + \theta(1 - 1/\psi)\mu + (\theta - 1)(\kappa_{y0} + \eta_{y0}(\kappa_{y1} - 1)) + (\kappa_{d0} + \eta_{d0}(\kappa_{d1} - 1)) \\ +\tfrac{1}{2}((\pi_{dy} - \gamma)^2\sigma_y^2 + ((\theta - 1)\kappa_{y1}\eta_{y1} + \kappa_{d1}\eta_{d1})^2\sigma_a^2 + \psi_d^2\sigma_y^2) = 0,\end{aligned} \tag{44}$$

$$\theta\omega^j + (\theta - 1)[(\tilde{\rho}\kappa_{y1} - 1)\eta_{y1} + \kappa_{y1}\eta_{y2}] + (\tilde{\rho}\kappa_{d1} - 1)\eta_{d1} + \kappa_{d1}\eta_{d2} = 0, \tag{45}$$

$$\theta + (\theta - 1)\eta_{y2} + \eta_{d2} + ((\theta - 1)\kappa_{y1}\eta_{y1} + \kappa_{d1}\eta_{d1})\rho_a = 0. \tag{46}$$

Equations (41)-(46), along with (15) and (16), form a system of 10 equations and 10 unknowns.

Given the model solution, we can solve for the risk free rate. The Euler equation implies

$$\begin{aligned} r_{f,t} &= -\log(E_t[\exp(m_{t+1})]) \\ &= -E_t[m_{t+1}] - \tfrac{1}{2}\operatorname{Var}_t[m_{t+1}], \end{aligned}$$

since the risk-free rate is known at time-t. The pricing kernel is given by

$$\begin{aligned}
m_{t+1} &= \theta\hat{\beta} + \theta(\omega^j \hat{a}_{t+1} - \hat{a}_t) - (\theta/\psi)\Delta\hat{y}_{t+1} + (\theta-1)\hat{r}_{y,t+1} \\
&= \theta\hat{\beta} + \theta(\omega^j \hat{a}_{t+1} - \hat{a}_t) - \gamma\Delta\hat{y}_{t+1} + (\theta-1)(\kappa_{y0} + \kappa_{y1}\hat{z}_{y,t+1} - \hat{z}_{y,t}) \\
&= \theta\hat{\beta} + (\theta-1)(\kappa_{y0} + \eta_{y0}(\kappa_{y1}-1)) - \gamma\mu + (\theta\omega^j + (\theta-1)[(\tilde{\rho}\kappa_{y1}-1)\eta_{y1} + \kappa_{y1}\eta_{y2}])\hat{a}_{t+1} \\
&\qquad - (\theta + (\theta-1)(\eta_{y2} + \kappa_{y1}\eta_{y1}\rho_a))\hat{a}_t + (\theta-1)\kappa_{y1}\eta_{y1}\sigma_a\varepsilon_{a,t+1} - \gamma\sigma_y\varepsilon_{y,t+1} \\
&= \theta\hat{\beta} + (\theta-1)(\kappa_{y0} + \eta_{y0}(\kappa_{y1}-1) + \kappa_{y1}\eta_{y1}\sigma_a\varepsilon_{a,t+1}) - \gamma\mu - \gamma\sigma_y\varepsilon_{y,t+1} + \omega^j\hat{a}_{t+1} - \hat{a}_t,
\end{aligned}$$

where the last line follows from imposing (42) and (43). Therefore, the risk-free rate is given by

$$r_{f,t} = \gamma\mu - \theta\hat{\beta} - (\theta-1)(\kappa_{y0} + \eta_{y0}(\kappa_{y1}-1)) - (\omega^j\hat{a}_{t+1} - \hat{a}_t) - \tfrac{1}{2}(\gamma\sigma_y)^2 - \tfrac{1}{2}((\theta-1)\kappa_{y1}\eta_{y1}\sigma_a)^2.$$

The unconditional expected risk-free rate is given by

$$E[r_f] = \gamma\mu - \theta\hat{\beta} - (\theta-1)(\kappa_{y0} + \eta_{y0}(\kappa_{y1}-1)) - \tfrac{1}{2}(\gamma\sigma_y)^2 - \tfrac{1}{2}((\theta-1)\kappa_{y1}\eta_{y1}\sigma_a)^2. \tag{47}$$

Plugging (41) into (47) implies

$$E[r_f] = -\hat{\beta} + \mu/\psi + \tfrac{1}{2}(\theta-1)\kappa_{y1}^2\eta_{y1}^2\sigma_a^2 + \tfrac{1}{2}((1/\psi-\gamma)(1-\gamma)-\gamma^2)\sigma_y^2. \tag{48}$$

We can also derive an expression for the equity premium. The Euler equation implies

$$\begin{aligned}
0 &= \log(E_t[\exp(\hat{m}_{t+1} + \hat{r}_{d,t+1})]) \\
&= E_t[\hat{m}_{t+1} + \hat{r}_{d,t+1}] + \tfrac{1}{2}\operatorname{Var}_t[\hat{m}_{t+1} + \hat{r}_{d,t+1}] \\
&= E_t[\hat{m}_{t+1}] + E_t[\hat{r}_{d,t+1}] + \tfrac{1}{2}(\operatorname{Var}_t[\hat{m}_{t+1}] + \operatorname{Var}_t[\hat{r}_{d,t+1}] + \operatorname{Cov}_t[\hat{m}_{t+1}, \hat{r}_{d,t+1}]) \\
\rightarrow \quad E_t[ep_{t+1}] &= E_t[r_{d,t+1} - r_{f,t}] = -\tfrac{1}{2}\operatorname{Var}_t[\hat{r}_{d,t+1}] - \operatorname{Cov}_t[\hat{m}_{t+1}, \hat{r}_{d,t+1}].
\end{aligned}$$

We first solve for the return on dividends, which is given by

$$\begin{aligned}
\hat{r}_{d,t+1} &= \kappa_{d0} + \kappa_{d1}\hat{z}_{d,t+1} - \hat{z}_{d,t} + \Delta\hat{d}_{t+1} \\
&= \kappa_{d0} + \kappa_{d1}(\eta_{d0} + \eta_{d1}\hat{a}_{t+2} + \eta_{d2}\hat{a}_{t+1}) - (\eta_{d0} + \eta_{d1}\hat{a}_{t+1} + \eta_{d2}\hat{a}_t) + \Delta\hat{d}_{t+1} \\
&= (\mu + \kappa_{d0} + (\kappa_{d1}-1)\eta_{d0}) + ((\tilde{\rho}\kappa_{d1}-1)\eta_{d1} + \kappa_{d1}\eta_{d2})\hat{a}_{t+1} - (\kappa_{d1}\eta_{d1}\rho_a + \eta_{d2})\hat{a}_t \\
&+ \kappa_{d1}\eta_{d1}\sigma_a\varepsilon_{a,t+1} + \pi_{dy}\sigma_y\varepsilon_{y,t+1} + \psi_d\sigma_y\varepsilon_{d,t+1}.
\end{aligned}$$

Therefore, the unconditional equity premium is given by

$$E[ep] = \tfrac{1}{2}(2\gamma - \pi_{dy})\pi_{dy}\sigma_y^2 - \tfrac{1}{2}\psi_d^2\sigma_y^2 - \tfrac{1}{2}(2(\theta-1)\kappa_{y1}\eta_{y1} + \kappa_{d1}\eta_{d1})\kappa_{d1}\eta_{d1}\sigma_a^2. \tag{49}$$

A.1 SPECIAL CASE 1 ($\sigma_a = \psi_d = 0$ & $\pi_{dy} = 1$) In this case, there is no valuation risk because $\hat{a}_t = 0$ and endowment/dividend risk is perfectly correlated ($\Delta\hat{y}_{t+1} = \Delta\hat{d}_{t+1} = \mu + \sigma\varepsilon_{t+1}$). Under these assumptions, it is easy to see that (48) and (49) reduce to (21) and (22) in the main text.

A.2 SPECIAL CASE 2 ($\sigma_y = 0$, $\rho_a = 0$, & $\sigma_a > 0$) In this case, there is no cash flow risk ($\Delta\hat{y}_{t+1} = \Delta\hat{d}_{t+1} = \mu$) and the preference shock follows a random walk ($\hat{a}_{t+2} = \hat{a}_{t+1} + \sigma_a\varepsilon^a_{t+1}$). Under these assumptions, the return on the endowment and dividend claims are identical so $\{\kappa_{y0}, \kappa_{y1}, \eta_{y0}, \eta_{y1}, \eta_{y2}\} = \{\kappa_{d0}, \kappa_{d1}, \eta_{d0}, \eta_{d1}, \eta_{d2}\} \equiv \{\kappa_0, \kappa_1, \eta_0, \eta_1, \eta_2\}$. Therefore, (48) and (49) reduce to (25) and (26) in the main text. Also, the exclusion restrictions simplify to

$$0 = \hat{\beta} + (1 - 1/\psi)\mu + (\kappa_0 + \eta_0(\kappa_1 - 1)) + \tfrac{\theta}{2}\kappa_1^2\eta_1^2\sigma_a^2, \tag{50}$$

$$0 = \omega^j + \eta_1(\kappa_1 - 1) + \eta_2\kappa_1, \tag{51}$$

$$\eta_2 = -1, \tag{52}$$

$$\kappa_0 = \log(1 + \exp(\eta_0)) - \kappa_1\eta_0, \tag{53}$$

$$\kappa_1 = \exp(\eta_0)/(1 + \exp(\eta_0)). \tag{54}$$

First, notice that $0 < \kappa_1 < 1$. Therefore, with AELR preferences ($\omega^{AELR} = 1$), it is easy to see that $\eta_1 = 1$. With DRT preferences ($\omega^{DRT} = \beta$), the solution for η_1 is more complicated. However, for $\psi = 1$, we guess and then verify that $\eta_1 = 0$. In this case, from (51) $\kappa_1 = \beta$ and (50) reduces to

$$0 = \log\beta + \kappa_0 - (1 - \beta)\eta_0, \tag{55}$$

This implies that $\eta_0 = \log\beta - \log(1 - \beta)$ and $\kappa_0 = -(1 - \beta)\log(1 - \beta) - \beta\log\beta$.

Unanticipated Valuation Risk With a small change in the timing of the valuation risk shock, we can derive a closed-form expression for the risk-free rate without relying on a Campbell-Shiller approximation to show that the asymptote is not due to the approximation. Building on special case 2, we assume $\Delta\hat{a}_{t+1} = \sigma\varepsilon^a_{t+1}$ instead of $\Delta\hat{a}_{t+1} = \sigma\varepsilon^a_t$, so $\Delta\hat{a}_{t+1}$ is no longer anticipated.

Preferences are given by (3), so the equilibrium condition that prices asset i is given by

$$1 = \beta E_t[(a_{t+1}/a_t)(c_{t+1}/c_t)^{-1/\psi}(V_{t+1}^{1-\gamma}/E_t[V_{t+1}^{1-\gamma}])^{1-1/\theta}r_{i,t+1}], \tag{56}$$

where we dropped the preference specific superscripts. We begin by conjecturing that the value function takes the form $V_t/c_t = \eta a_t^{\theta/(1-\gamma)}$. Substituting the guess into the value function implies

$$\eta = \frac{1 - \beta}{1 - \beta\exp((1 - \gamma)\mu + \sigma^2/2)}, \tag{57}$$

which verifies our conjecture. Therefore, we can substitute into (56) to obtain

$$\begin{aligned}\hat{r}_{f,t} &= -\log\beta + \mu/\psi - \log(E_t[(a_{t+1}/a_t)^\theta])/\theta,\\ &= -\log\beta + \mu/\psi - \log(E_t[\exp(\theta\Delta\hat{a}_{t+1})])/\theta,\\ &= -\log\beta + \mu/\psi - \theta\sigma^2/2.\end{aligned}$$

This result shows that the risk-free rate will inherit the asymptote in θ.

B ESTIMATION METHOD

We use the dataset provided by Albuquerque et al. (2016), which contains annual observations from 1929 to 2011 of U.S. per-capita real consumption, the real market log return, the risk-free rate, per-capita real dividends, and the log price-dividend ratio. The estimation method is conducted in two stages. The first stage estimates key moments in the data using a 2-step Generalized Method of Moments (GMM) estimator with a Newey and West (1987) weighting matrix with 10 lags.

The second stage implements a Simulated Method of Moments (SMM) procedure that searches for a parameter vector that minimizes the distance between the GMM estimates in the data and the short-sample predictions of the model, weighted by the GMM estimate of the covariance matrix. The following steps outline the complete estimation method:

1. Use GMM to estimate the data moments, $\tilde{\Psi}_D$, and the corresponding covariance matrix, $\tilde{\Sigma}_D$.
2. Specify a guess, $\hat{\theta}_0$, for the N_e estimates parameters, $\theta \equiv [\gamma, \psi, \beta, \sigma_y, \mu, \psi_d, \pi_{dy}, \sigma_a, \rho_a]'$, and the parameter covariance matrix, Σ_P, which is initialized as a diagonal matrix.
3. Use simulated annealing to minimize the distance between the data and model moments.
   (a) For all $i \in \{0, \dots, N_d\}$, perform the following steps:
      i. Draw a candidate vector of parameters, $\hat{\theta}_i^{cand}$, where

$$\hat{\theta}_i^{cand} \sim \begin{cases}\hat{\theta}_0 & \text{for } i = 0,\\ \mathbb{N}(\hat{\theta}_{i-1}, c\Sigma_P) & \text{for } i > 0.\end{cases}$$

      We set c to target an overall acceptance rate of roughly 30%.
      ii. Solve the Campbell-Shiller approximation of the model given $\hat{\theta}_i^{cand}$.
      iii. Simulate the monthly model $1{,}000$ times for the same length as the data plus a burn-in period. We burn off $10{,}000$ months so that the first period closely approximates a representative draw from the model's ergodic distribution. For each simulation j, calculate the moments, $\Psi_{M,j}(\hat{\theta}_i^{cand})$, analogous to those in the data.

iv. Calculate the median moments across the short-sample simulations, $\bar{\Psi}_M(\hat{\theta}_i^{cand}) = \text{median}\left(\{\Psi_{M,j}(\hat{\theta}_i^{cand})\}_{j=1}^{1000}\right)$, and evaluate the objective function,

$$J_i^{cand} = [\bar{\Psi}_M(\hat{\theta}_i^{cand}) - \tilde{\Psi}_D]'\tilde{\Sigma}_D^{-1}[\bar{\Psi}_M(\hat{\theta}_i^{cand}) - \tilde{\Psi}_D]$$

v. Accept or reject the candidate draw according to

$$(\hat{\theta}_i, J_i) = \begin{cases} (\hat{\theta}_i^{cand}, J_i^{cand}) & \text{if } i = 0, \\ (\hat{\theta}_i^{cand}, J_i^{cand}) & \text{if } \min(1, \exp(J_{i-1} - J_i^{cand})/t) > \hat{u}, \\ (\hat{\theta}_{i-1}, J_{i-1}) & \text{otherwise,} \end{cases}$$

where $t \leq 1$ is the temperature and $\hat{u}$ is a draw from a uniform distribution. The lower the temperature, the more likely it is that the candidate draw is rejected.

(b) Find the parameter draw $\hat{\theta}^{min}$ that corresponds to J^{min}, and update Σ_P.

i. Discard the first $N_d/2$ draws. Stack the remaining draws in a $N_d/2 \times N_e$ matrix, $\hat{\Theta}$, and define $\tilde{\Theta} = \hat{\Theta} - \sum_{i=1}^{N_d/2} \hat{\theta}_{i,j}/N_{m,sub}$.

ii. Calculate $\Sigma_P^{up} = \tilde{\Theta}'\tilde{\Theta}/N_{m,sub}$.

4. Repeat the previous step, initializing at draw $\hat{\theta}_0 = \hat{\theta}^{min}$, covariance matrix $\Sigma_P = \Sigma_P^{up}$, and gradually decreasing t each time, until J^{min} does not decrease more than some tolerance.