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Macroeconomic Responses to Uncertainty Shocks: The Perils of Recursive Orderings[*]

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Abstract

A common practice in empirical macroeconomics is to examine alternative recursive orderings of the variables in structural vector autoregressive (VAR) models. When the implied impulse responses look similar, the estimates are considered trustworthy. When they do not, the estimates are used to bound the true response without directly addressing the identification challenge. A leading example of this practice is the literature on the effects of uncertainty shocks on economic activity. We prove by counterexample that this practice is invalid in general, whether the data generating process is a structural VAR model or a dynamic stochastic general equilibrium model.

Keywords: Cholesky decomposition; orthogonalization; simultaneity; endogeneity; uncertainty; business cycle

JEL Classifications: C32, C51, E32

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1 INTRODUCTION

Structural vector autoregressive (VAR) analysis continues to be widely applied in empirical macroeconomics. In recent years there has been a proliferation of research on new approaches to the identification of structural VAR models; yet, many applied researchers continue to use much simpler identification strategies that impose a recursive structure on the structural impact multiplier matrix. This allows this matrix to be estimated by applying a lower triangular Cholesky decomposition to the reduced-form VAR residual covariance matrix with the diagonal elements normalized to be positive. Sometimes this approach can be justified on economic grounds, but more often it is used as a means of creating mutually uncorrelated shocks that have no obvious structural interpretation.

The fact that recursive VAR models in general will not recover the population responses when the true model is not recursive has been discussed at length in the econometrics literature (see, e.g., Braun and Mittnik, 1993; Cooley and Leroy, 1985; Kilian and Lütkepohl, 2017; Leamer, 1985). However, this fact has done little to diminish the appeal of ad hoc recursive orderings in applied work, in particular when relating one macroeconomic model variable to another without reference to an explicit structural model. Even having acknowledged the well-known limitations of recursive models, many applied users believe that ad hoc recursive orderings may be used to learn about the quantitative importance of causal effects in the data. A leading example is the literature on the effects of uncertainty shocks on economic activity. For example, Altig et al. (2020) concedes that "drawing causal inferences from VARs is challenging—in part because policy, and policy uncertainty, can respond to current and anticipated future economic conditions", but argues that "despite the challenges, [recursively identified] VARs are useful for...gauging whether uncertainty innovations foreshadow weaker macroeconomic performance conditional on standard macro and policy variables." This view is widely shared in the literature.[1]

A common practice even in top economics journals has been to report impulse response estimates based on alternative orderings of the model variables with the uncertainty measure ordered

[1]Recent exceptions are Berger et al. (2020), Ludvigson et al. (2021), and Carriero et al. (2021) who propose alternative VAR approaches to identifying uncertainty shocks.

first or last. Evidence that the responses are robust to alternative orderings is taken as evidence that the shock of interest has been identified and the impulse response estimates are trustworthy. Another closely related argument is that alternative recursive orderings may be used to bound the true impulse response without directly addressing the identification challenge (e.g., Altig et al., 2020; Bachmann et al., 2013; Baker et al., 2016; Basu and Bundick, 2017; Bloom, 2009; Caggiano et al., 2014; Caldara and Iacoviello, 2022; Fernández-Villaverde et al., 2015; Leduc and Liu, 2016). The validity of these popular arguments has never been formally examined. In this paper, we prove by counterexample that both arguments are incorrect, calling into question many empirical results reported in the literature. Our examples are designed to persuade researchers of the dangers of relying on ad hoc recursive orderings and to dispel common misconceptions found in applied work.

What is new about our analysis is not the insight that recursively identified VAR models are questionable if their identifying restrictions are not supported by extraneous evidence. Rather what we add to the literature is the result that the robustness of impulse responses to alternative ad hoc recursive orderings does not establish that the true response has been identified. Nor can alternative recursive orderings be used to bound the true responses, as commonly argued in applied work. We show this analytically based on structural VAR examples and by simulation using data generated from dynamic stochastic general equilibrium (DSGE) models.

The remainder of the paper is organized as follows. After reviewing examples of economically motivated fully recursive and block recursive VAR models in Section 2, we turn to the common practice of conducting robustness checks based on alternative recursive orderings in Section 3. This section also examines the practice of constructing upper and lower bounds on the effects of uncertainty shocks based on alternatively ordering the uncertainty variable first and last in the model without directly addressing the identification challenge. We analytically demonstrate the invalidity of both approaches using stylized VAR processes. Our analysis is not subject to reduced-form model misspecification or estimation error, which facilitates the comparison of alternative models.

Ad hoc recursive VAR orderings fail in particular when real activity and uncertainty are simultaneously determined. This simultaneity is best illustrated within the context of a fully specified economic model. In Section 4, we examine the credibility of recursive VAR models when the data are generated by calibrated DSGE models of the determination of aggregate uncertainty. We first draw attention to the fact that recursive VAR models of the effects of aggregate uncertainty shocks are invalid, regardless of the ordering, when uncertainty is fully endogenous in the data generating process. In this environment, the uncertainty shock the VAR model seeks to identify does not exist, which immediately invalidates the empirical model. We then show that the recursive VAR is also unable to recover the population response of output to an uncertainty shock when aggregate uncertainty is only partially endogenous. We also confirm that alternative recursive orderings fail to bound the population response in general. Thus, our analysis of DSGE models supports and reinforces our earlier conclusion that recursive VAR models of the effects of uncertainty shocks cannot be trusted. The concluding remarks are in Section 5.

2 THE CASE FOR RECURSIVELY IDENTIFIED STRUCTURAL VAR MODELS

There are situations in which a recursive model can be economically justified, but such situations are rare in applied work.[2] More often, researchers rely on a block recursive model structure. For example, by ordering a narrative measure of exogenous monetary or fiscal policy shocks first in the model, we can identify the causal effect of these shocks on the remaining model variables without taking a stand on the identification of the other structural shocks. Since the responses of interest are invariant to the identification of the remaining structural shocks, one may estimate such models without loss of generality based on a Cholesky decomposition, retaining only the responses to the first structural shock. The same approach works when the variable ordered first is known to be predetermined with respect to the other variables based on extraneous information (e.g., Kilian and Zhou, 2022).

[2]One example is the fully recursive structural model of the global oil market in Kilian (2009). The exclusion restrictions in this model in part can be derived from extraneous evidence on the magnitude of the one-month price elasticity of oil supply (Kilian, 2022) and in part are implied by the construction of the measure of global real activity in that model (Kilian and Zhou, 2018).

Similarly, semi-structural models of monetary policy, as first proposed by Bernanke and Blinder (1992), involve ordering the policy instrument last in the structural model, which allows the monetary policy shock to be interpreted as the structural shock to a policy rule, according to which policy responds contemporaneously to variation in observables ordered above the policy instrument. The policy shock reflects by construction departures from the expected setting of the policy instrument. The latter class of models is also block recursive, but the order of the blocks is reversed with the structural shocks in the upper block remaining unidentified. As in the previous examples, the model can be estimated without loss of generality based on a Cholesky decomposition of the reduced-form error covariance matrix, as long as we disregard all responses to shocks other than the monetary policy shock. In all these examples, the use of a Cholesky decomposition is justified, provided there is evidence for the exclusion restrictions identifying the structural shock of interest.

3 ROBUSTNESS CHECKS FOR ALTERNATIVE AD HOC RECURSIVE ORDERINGS

In general, the response estimates implied by a Cholesky decomposition will differ depending on the ordering of the variables in the structural VAR model. This is not surprising since each of these recursive models assumes a different economic structure. In an n-dimensional VAR process, there are $n!$ alternative Cholesky orderings. Each of these orderings will imply different impulse response estimates unless the covariances of the reduced-form residuals are zero, but, for low enough error correlations, the responses will tend to be similar across alternative orderings.

In a model that is explicitly identified based on economic reasoning, as in the examples in Section 2, we know a priori that one of these orderings is correct and the others are not, so the response estimates are unique. This is not the case when Cholesky decompositions are used in an ad hoc fashion without explicit economic motivation, as is often the case in applied work. It is the latter situation that this paper addresses.

A classical example is the literature on the effect of uncertainty shocks on economic activity. This question has given rise to a large literature relying on recursively identified structural VAR models (e.g., Altig et al., 2020; Bachmann et al., 2013; Baker et al., 2016; Basu and Bundick, 2017;

Bekaert et al., 2013; Bloom, 2009; Caggiano et al., 2014; Caldara and Iacoviello, 2022; Fernández-Villaverde et al., 2015; Jurado et al., 2015; Leduc and Liu, 2016). The empirical response estimates in turn have stimulated theoretical work on the transmission of uncertainty shocks.

Aggregate uncertainty in this literature is typically measured by the volatility of macroeconomic or financial aggregates. As noted by Ludvigson et al. (2021), empirical studies often differ according to whether aggregate uncertainty is ordered ahead of or after real activity variables in the VAR model. It is useful to analyze this situation in a bivariate setting. Allowing for additional variables ordered in between the first and the last variable does not change the logic of the arguments below. For example, consider a stylized model of the impact of uncertainty shocks on real GDP growth, where the reduced-form shocks, u_t, are linked to the structural shocks, w_t, through the structural impact multiplier matrix, B_0^{-1} with elements $b_0^{ij} = \partial u_{i,t}/\partial w_{j,t}$, $i \in \{1,2\}$, $j \in \{1,2\}$.[3] There are two recursive models of this relationship. We can postulate that

$$\begin{pmatrix} u_t^{\Delta gdp} \\ u_t^{\text{uncertainty}} \end{pmatrix} = \begin{bmatrix} b_0^{11} & 0 \\ b_0^{21} & b_0^{22} \end{bmatrix} \begin{pmatrix} w_t^{\text{other}} \\ w_t^{\text{uncertainty}} \end{pmatrix} \tag{1}$$

or, alternatively, that

$$\begin{pmatrix} u_t^{\text{uncertainty}} \\ u_t^{\Delta gdp} \end{pmatrix} = \begin{bmatrix} b_0^{11} & 0 \\ b_0^{21} & b_0^{22} \end{bmatrix} \begin{pmatrix} w_t^{\text{uncertainty}} \\ w_t^{\text{other}} \end{pmatrix}. \tag{2}$$

A key difference from the recursive VAR examples in Section 2 is that in the current setting there is no compelling reason to prefer one ordering over the other. On the one hand, it has been noted that changes in economic activity may affect the uncertainty about the economy, which suggests ordering uncertainty last. On the other hand, it seems reasonable to expect positive uncertainty shocks to affect economic activity within the current month, which argues for ordering uncertainty first. Neither recursive model is consistent with both economic arguments, suggesting that at least one, if not both, of these models are misspecified.

[3]As is standard, all shocks are zero in expectation and the variance-covariance matrix of w_t is normalized to an identity matrix.

Many studies in this literature seek to resolve this concern by reporting impulse response estimates based on a recursive model in which a direct measure of economic uncertainty is ordered first as well as estimates based on an alternative recursive model in which this uncertainty measure is ordered last. If the response estimates are similar across these specifications, this is taken as evidence that response estimates are robust and hence trustworthy. It is this practice that we discuss next.

3.1 ROBUSTNESS TO ALTERNATIVE ORDERINGS DOES NOT MEAN THE TRUE RESPONSE IS IDENTIFIED

In general, a finding that the impulse responses to uncertainty shocks are invariant to the ordering would not be expected. The responses will be identical only when the reduced-form error correlation is zero. Even in that case, however, examining alternative recursive orderings is not sufficient. It is entirely possible and indeed plausible that the underlying population model is a simultaneous equations model that allows uncertainty to respond endogenously to economic activity conditional on past data. In fact, the recent literature provides many arguments why uncertainty is simultaneously determined with economic activity rather than determined recursively. For some choices of the model parameters, such a simultaneous equations model may imply zero correlation in the reduced-form errors, yet the responses in the population model need not look anything like the responses from recursively identified models. For example, consider the population model

$$\begin{aligned} \begin{pmatrix} u_t^{\Delta gdp} \\ u_t^{\text{uncertainty}} \end{pmatrix} &= \begin{bmatrix} 1 & 0.5 \\ -0.5 & 1 \end{bmatrix} \begin{pmatrix} w_t^{\text{other}} \\ w_t^{\text{uncertainty}} \end{pmatrix}, \\ \Sigma = B_0^{-1}(B_0^{-1})' = \begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix}, &\quad \text{chol}(\Sigma) = \begin{bmatrix} 1.118 & 0 \\ 0 & 1.118 \end{bmatrix}, \end{aligned} \tag{3}$$

which has uncorrelated reduced-form errors.[4] In this model, positive shocks to uncertainty increase economic growth on impact, consistent with "growth options" theories, whereas positive shocks to

[4]Uncorrelated reduced-form errors may arise when the off-diagonal elements of the structural impact multiplier matrix are of different signs. When both structural shocks have the same variance, the reduced-form errors will be uncorrelated when the off-diagonal elements are of the same magnitude in absolute terms. More generally, the off-diagonal elements can differ in absolute value. For example, $B_0^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0.75 & 1.5 \end{bmatrix}$ also implies mutually uncorrelated reduced-form errors.

economic activity reduce economic uncertainty in line with the analysis in Ludvigson et al. (2021).

Model (3) is a counterexample to the notion that, when Σ is diagonal, B_0 and B_0^{-1} must also be diagonal. The population impact response of real GDP growth to a structural uncertainty shock in this model is 0.5. Imposing a recursive ordering with real GDP growth ordered first implies an impact response of real GDP growth to the uncertainty shock that is 0. After re-ordering the two variables such that uncertainty is ordered first, the Cholesky estimate yields the same impact response of 0 for real GDP growth in response to an uncertainty shock. An applied user thus would be tempted to conclude that the robust finding of a zero response for both orderings means that the population response must be zero, when in reality the population response is 0.5. In short, the response being robust within the universe of recursive orderings does not mean that it is valid when allowing the true model to be nonrecursive.

3.2 Alternative recursive orderings do not bound the population response

There is no particular economic reason for the population process to imply reduced-form error correlations of zero. In the more typical case of non-zero correlations, we would expect the orthogonalized responses based on models (1) and (2) to differ from each other and from the population responses. This fact has not dissuaded practitioners from reporting response estimates based on alternative recursive orderings.

A common argument in the literature is that, when VAR studies order uncertainty before output, the authors are not necessarily claiming that uncertainty is exogenous, but they are simply conditioning on this hypothesis for the purposes of illustration. The belief is that this assumption yields an upper bound on the effect of uncertainty shocks (see, e.g., Caggiano et al., 2014). Similarly, by ordering uncertainty last, this argument goes, we give all other shocks a chance to explain the data first and hence end up with a lower bound on the effect of uncertainty shocks. For example, Jurado et al. (2015) suggest that "as uncertainty is placed last in the VAR, the effects of uncertainty shocks on the other variables in the system are measured after we have removed all the variation in uncertainty that is attributable to shocks to the other endogenous variables in the system. That the effects of uncertainty shocks are still non-trivial is consistent with the view that uncertainty has important implications for economic activity."

As shown next, this approach is not valid. It is not possible in general to bound the population effect by considering alternative orderings. For example, consider the population process:

$$\begin{pmatrix} u_t^{\Delta gdp} \\ u_t^{\text{uncertainty}} \end{pmatrix} = \begin{bmatrix} 1 & 0.5 \\ -0.9 & 1 \end{bmatrix} \begin{pmatrix} w_t^{\text{other}} \\ w_t^{\text{uncertainty}} \end{pmatrix},$$
$$\Sigma = B_0^{-1}(B_0^{-1})' = \begin{bmatrix} 1.25 & -0.4 \\ -0.4 & 1.81 \end{bmatrix}, \quad \text{chol}(\Sigma) = \begin{bmatrix} 1.118 & 0 \\ -0.3578 & 1.2969 \end{bmatrix}. \tag{4}$$

In this case, the impact response of real GDP growth is 0 compared to the population response of 0.5. Reversing the order of the variables yields

$$\begin{pmatrix} u_t^{\text{uncertainty}} \\ u_t^{\Delta gdp} \end{pmatrix} = \begin{bmatrix} 1 & -0.9 \\ 0.5 & 1 \end{bmatrix} \begin{pmatrix} w_t^{\text{uncertainty}} \\ w_t^{\text{other}} \end{pmatrix},$$
$$\Sigma = B_0^{-1}(B_0^{-1})' = \begin{bmatrix} 1.81 & -0.4 \\ -0.4 & 1.25 \end{bmatrix}, \quad \text{chol}(\Sigma) = \begin{bmatrix} 1.3454 & 0 \\ -0.2973 & 1.0778 \end{bmatrix}. \tag{4$'$}$$

The impact response of real GDP growth is now -0.2973. The conventional wisdom would suggest that the true response must be between -0.2973 and 0, but this is obviously not the case because the population response is 0.5 in this model, which is of the opposite sign of the range of values bounded by the recursive estimates. In fact, a similarly erroneous conclusion would have been reached in Model (3), which would have suggested that the population response is bounded from above and below by 0, when it actually is 0.5. These two examples illustrate that the population response cannot be bounded based on responses obtained under alternative orderings.

This is not the only problem with the argument that we can learn about the importance of uncertainty shocks by extracting the orthogonal component of the uncertainty shock when controlling for real GDP growth by ordering the uncertainty variable last, as in model (1). If the resulting uncertainty shocks have large effects on real activity, according to this argument, we can be confident that the true effect is at least as large in absolute value as the estimated effect, even in the absence of formal identification. As we have shown, however, when ordering uncertainty last, the impact response of real GDP growth to an exogenous uncertainty shock is zero by construction. Proponents of this approach would therefore conclude that, given a bound of zero, the impact response must be zero, positive or negative. This is something we already knew even before estimating the model, raising the question of what value this approach adds.[5]

The point of these examples is to rigorously demonstrate that there is no reason to expect any of the recursive estimates to recover the population response when the population process is not recursive. Nor is there a reason for recursive estimates to provide an approximation to the population response in general. This is true whether the reduced-form errors are positively correlated, uncorrelated, or negatively correlated, as is typically the case in the empirical literature on uncertainty shocks (see, e.g., Bernstein et al., 2022). It is also true whether alternative orderings produce the same response or not.

4 EXAMINING AD HOC RECURSIVE IDENTIFICATION THROUGH DSGE MODELS

The stylized examples discussed in this paper are simple enough to facilitate closed-form solutions. Unlike in simulation studies, these results are not subject to estimation error. Nor are they subject to model misspecification error. They demonstrate clearly what the issues are with relying on ad hoc recursive orderings. In this section, we employ a DSGE model of the determination of aggregate uncertainty to further examine the conditions under which one would expect ad hoc recursive VAR models to fail. Our analysis sheds light on the origin of the simultaneity that invalidates the use of ad hoc recursive orderings.

4.1 DSGE MODEL For expository purposes, we use a textbook real business cycle model augmented to include disaster risk (Gourio, 2012) and recursive preferences (Epstein and Zin, 1989). Disaster risk introduces nonlinearity into the decisions rules, while recursive preferences sepa-

[5] One may object that this fact is irrelevant if researchers are interested in the peak response rather than the impact response. We will return to this point in Section 4 and show that even in that case ordering uncertainty last does not provide a valid bound.

rate risk aversion from the intertemporal elasticity of substitution. These features determine the response of output to uncertainty shocks and generate endogenous fluctuations in uncertainty.

The representative household solves the Bellman equation

$$J(b_t) = \max_{c_t, n_t, b_{t+1}} \left[(1-\beta) u_t^{1-1/\psi} + \beta (E_t[J(b_{t+1})^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}}$$

subject to

$$u_t = c_t^{\eta} (1-n_t)^{1-\eta},$$

$$c_t + b_{t+1}/r_t = w_t n_t + b_t + d_t,$$

where $\beta \in (0,1)$ is the discount factor, $\gamma \geq 0$ determines risk aversion, $\psi \geq 0$ is the intertemporal elasticity of substitution, u_t is the utility function, c_t is consumption, n_t is labor hours, b_t is a risk-free bond with return r_t, w_t is the wage rate, and d_t are lump-sum dividends from the firm. The term $z_t \equiv (E_t[J_{t+1}^{1-\gamma}])^{\frac{1}{1-\gamma}}$ is the risk-adjusted expectation operator.

The representative firm solves the Bellman equation

$$V(k_t) = \max_{n_t, i_t, k_{t+1}} d_t + E_t[x_{t+1} V(k_{t+1})]$$

subject to

$$d_t = y_t - w_t n_t - i_t,$$

$$y_t = a_t k_t^{\alpha} n_t^{1-\alpha},$$

$$k_{t+1} = \Theta_{t+1}((1-\delta) k_t + i_t),$$

where x_{t+1} is the pricing kernel derived from the household's optimality conditions, y_t is output, i_t is investment, k_t is the stock of capital that depreciates at rate δ, and a_t is the technology state, which follows

$$\ln a_t = \rho_a \ln a_{t-1} + \sigma_{a,t-1} \varepsilon_{a,t}, \;\; -1 < \rho_a < 1, \; \varepsilon_{a,t} \sim \mathbb{N}(0,1), \tag{5}$$

$$\ln \sigma_{a,t} = (1-\rho_{av}) \ln \bar{\sigma}_a + \rho_{av} \ln \sigma_{a,t-1} + \sigma_{av} \varepsilon_{av,t}, \;\; -1 < \rho_{av} < 1, \; \varepsilon_{av,t} \sim \mathbb{N}(0,1). \tag{6}$$

Parameters	Value	Parameters	Value
Discount Factor (β)	0.995	Risk Aversion (γ)	80
Cost Share of Capital (α)	0.333	Intertemporal Elasticity of Substitution (ψ)	1
Capital Depreciation Rate (δ)	0.025	Frisch Elasticity of Labor Supply (η^λ)	2
Size of Disaster (θ)	0.95	Disaster Risk Threshold (e^*)	0.97
Disaster Risk AC (ρ_e)	0.90	Disaster Risk Shock SD ($\bar{\sigma}_e$)	0.0075
Technology AC (ρ_a)	0.90	Technology Shock SD ($\bar{\sigma}_a$)	0.007
Disaster Risk Vol. Shock AC (ρ_{ev})	0.90	Disaster Risk Vol. Shock SD (σ_{ev})	0.05
Technology Vol. Shock AC (ρ_{av})	0.90	Technology Vol. Shock SD (σ_{av})	0.05

Table 1: DSGE model calibration at quarterly frequency.

Following Gourio (2012), Θ_t is a capital quality (or depreciation) shock that is determined by

$$\Theta_t = \mathbb{I}(e_t \geq e^*) + \theta\mathbb{I}(e_t < e^*), \tag{7}$$

where $\mathbb{I}(\cdot)$ is an indicator function. When $e_t \geq e^*$, there is no capital quality loss, so $\Theta_t = 1$. When $e_t < e^*$, a disaster causes capital quality loss, so $\Theta_t = \theta < 1$. The likelihood of a disaster, e_t, evolves according to

$$\ln e_t = \rho_e \ln e_{t-1} + \sigma_{e,t-1}\varepsilon_{e,t}, \ \ -1 < \rho_e < 1, \ \varepsilon_{e,t} \sim \mathbb{N}(0,1), \tag{8}$$

$$\ln \sigma_{e,t} = (1-\rho_{ev})\ln \bar{\sigma}_e + \rho_{ev} \ln \sigma_{e,t-1} + \sigma_{ev}\varepsilon_{ev,t}, \ \ -1 < \rho_{ev} < 1, \ \varepsilon_{ev,t} \sim \mathbb{N}(0,1). \tag{9}$$

The online appendix derives the first-order conditions and defines the competitive equilibrium.

We follow Bernstein et al. (2022) and define aggregate uncertainty as the uncertainty about log output growth,

$$\mathcal{U}_t = \sqrt{E_t[(\ln(y_{t+1}/y_t) - E_t[\ln(y_{t+1}/y_t)])^2]}. \tag{10}$$

This definition is equivalent to the uncertainty surrounding the level of log output because y_t is known at time t and cancels from the definition of uncertainty in (10). $\mathcal{U}_t$ can endogenously fluctuate over time due to the nonlinear propagation of the level shocks, and it can exogenously fluctuate over time due to the volatility shocks.

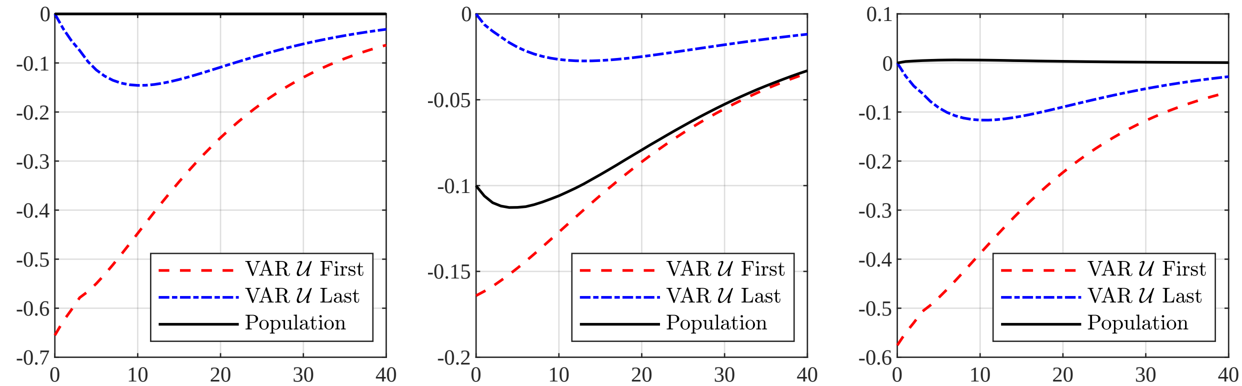
Table 1 summarizes the parameter values. All of the deep parameters are set to values common in the literature. We set the disaster risk threshold, e^*, to 0.97. The size of the disaster, θ, is set to 0.95, so 5% of the capital stock is lost when the disaster state occurs. For the shock processes, we set the persistence parameters to 0.9, and we set the shock standard deviations so the model generates output volatility that is consistent with the data.[6]

We consider different shock specifications to highlight the issues that can arise with ad hoc recursive VAR models. For each specification, we generate time series for log output and output uncertainty of length $T = 1{,}000{,}000$ and construct VAR(4) response estimates from alternative recursive orderings. For such large T, this approach provides a close approximation to the asymptotic limit of the VAR impulse response functions. Qualitatively similar results are obtained when including more lags in the VAR model, as suggested by the literature on approximating impulse responses in VAR(∞) processes (e.g., Inoue and Kilian, 2002).

4.2 FULLY ENDOGENOUS UNCERTAINTY One of the assumptions in both recursive and nonrecursive VAR models of the transmission of uncertainty shocks is that there exists a second-moment (or volatility) shock in the underlying data generating process. However, one possible scenario is that uncertainty is fully endogenous in the data generating process, in which case fluctuations in uncertainty are entirely explained by the first moment shocks. In this case, recursive VAR models of the effects of aggregate uncertainty shocks are invalid regardless of the ordering, because the uncertainty shock the VAR model seeks to identify does not exist. This concern also applies to VAR models based on nonstandard identification approaches in which uncertainty is allowed to be determined simultaneously with real activity (e.g., Carriero et al., 2021; Ludvigson et al., 2021).

For example, suppose the second-moment shocks to technology and disaster risk are shut down, so the model dynamics are only driven by the two first-moment shocks. Figure 1a reports the responses of output to uncertainty shocks based on the two alternative recursive VAR models. All responses have been normalized such that the uncertainty shock is of the same magnitude. Based

[6] We solve the nonlinear model globally using the policy function iteration algorithm described in Richter et al. (2014) based on the theoretical work in Coleman (1991). Conditional on satisfying the equilibrium system, the algorithm minimizes the Euler equation errors on each node in the state space and computes the maximum change in the policy functions. It then iterates until the maximum change is below a specified tolerance criterion. The online appendix describes the solution method in more detail.



(a) Shocks: $\varepsilon_a, \varepsilon_e$ (b) Shocks: $\varepsilon_e, \varepsilon_{ev}$ (c) Shocks: $\varepsilon_e, \varepsilon_a, \varepsilon_{av}$

Figure 1: Output responses to an uncertainty shock.

on these responses, one would conclude that uncertainty shocks negatively affect output. The effects are particularly large when uncertainty is ordered first in the recursive VAR. They are still non-negligible even when uncertainty is ordered last. However, in this model both recursive orderings yield inconsistent response estimates. What the VAR model is effectively doing is capturing shocks that are linear combinations of the level shocks in the DSGE model.

4.3 PARTIALLY ENDOGENOUS UNCERTAINTY Now suppose there exists an exogenous uncertainty shock in the data generating process. As a first pass, we postulate that there are level and volatility shocks to disaster risk and no technology shocks in the DSGE model. In this environment, using a recursive VAR is still invalid because both shocks affect output and aggregate uncertainty on impact.

Figure 1b illustrates the asymptotic bias of the VAR estimates of the response of output to an exogenous uncertainty shock. The population response indicates that an uncertainty shock causes a 10 basis point drop in output on impact. The impact response in the recursive VAR is over-estimated when uncertainty is ordered first, but under-estimated when uncertainty is ordered last, showing that both recursive orderings fail to recover the population response. The reason is that the VAR shocks are linear combinations of the level and volatility shocks in the DSGE model.

One might conclude from this example that the responses from the two recursive VARs may be used to bound the impact of uncertainty on output, but this result is not guaranteed. To illustrate this point, Figure 1c shows the responses when the DSGE model contains a level shock to disaster risk and both level and volatility shocks to technology. The population response shows that a positive uncertainty shock causes a small increase in output. However, the responses from both recursive VARs indicate a decline in output. The response is particularly large when uncertainty is ordered first, in which case output falls on impact by nearly 60 basis points. This result is robust to including a third variable in the VAR model to account for the presence of three structural shocks in the DSGE model. Likewise, the recursive approach would fail if we allowed for two distinct volatility shocks in the DSGE model. We conclude that recursively identified VARs based on simulated data from calibrated macro models with endogenous uncertainty do not robustly identify uncertainty shocks, mirroring the conclusion in Section 3.

5 CONCLUSION

The common practice of reporting estimates from alternative recursive identification schemes and verifying the robustness of the conclusions is misleading. Robustness within the class of recursive models does not ensure the validity of the responses when the population model is non-recursive. There is also no support for the belief that recursive VARs are useful for gauging whether uncertainty innovations foreshadow weaker or stronger macroeconomic performance conditional on other variables. In particular, there is no support for the notion that we can learn about the importance of uncertainty shocks by extracting the orthogonal component of the uncertainty shock, controlling for other variables such as real GDP growth. Nor is the practice of bounding response estimates based on alternative orderings justified.

While we made these points in the context of one of the leading applications of recursively identified models in the literature, our analysis transcends this illustrative example and applies more generally to other VAR applications as well. One example is the literature on modeling the link between bond yields (and more generally financial conditions) and real activity (e.g., Gilchrist et al., 2009). Another example is the literature on modeling the link between interest rates and forward-looking indicators such as commodity prices or stock prices (e.g., Hanson, 2004; Sims, 1992). In fact, the points we are making are not specific to structural VAR models. Analogous identification issues also arise in empirical analysis based on local projections.

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Online Appendix to:
Macroeconomic Responses to Uncertainty Shocks: The Perils of Recursive Orderings*

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A DSGE MODEL DERIVATIONS

Representative Household The representative household solves the Bellman equation

$$J(b_t) = \max_{c_t, n_t, b_{t+1}} \left[(1-\beta) u_t^{1-1/\psi} + \beta (E_t[J(b_{t+1})^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}} \tag{1}$$

subject to

$$u_t = c_t^{\eta}(1-n_t)^{1-\eta}, \tag{2}$$

$$c_t + b_{t+1}/r_t = w_t n_t + b_t + d_t. \tag{3}$$

The first order conditions are given by

$$\begin{aligned} c_t: \quad \eta(1-\beta) J_t^{1/\psi} u_t^{1-1/\psi}/c_t &= \lambda_t, \\ n_t: \quad (1-\eta)(1-\beta) J_t^{1/\psi} u_t^{1-1/\psi}/(1-n_t) &= \lambda_t w_t, \\ \beta J_t^{1/\psi} (E_t[J_{t+1}^{1-\gamma}])^{\frac{\gamma-1/\psi}{1-\gamma}} E_t[J_{t+1}^{-\gamma} J_{b,t+1}] &= \lambda_t / r_t. \end{aligned}$$

The envelope condition implies

$$J_{b,t} = \lambda_t.$$

Combining these results implies

$$w_t = (1-\eta)c_t/(\eta(1-n_t)), \tag{4}$$

$$1 = E_t[x_{t+1} r_t], \tag{5}$$

$$x_{t+1} = \beta(c_t/c_{t+1})(u_{t+1}/u_t)^{1-1/\psi}(J_{t+1}/z_t)^{1/\psi-\gamma}. \tag{6}$$

Representative Firm The representative firm solves the Bellman equation

$$V(k_t) = \max_{n_t, k_{t+1}, i_t} d_t + E_t[x_{t+1} V(k_{t+1})]$$

subject to

$$d_t = y_t - w_t n_t - i_t,$$

$$y_t = a_t k_t^{\alpha} n_t^{1-\alpha}, \tag{7}$$

$$k_{t+1} = \Theta_{t+1}((1-\delta)k_t + i_t). \tag{8}$$

Substituting in all constraints yields

$$V(k_t) = \max_{n_t, i_t} a_t k_t^{\alpha} n_t^{1-\alpha} - w_t n_t - i_t + E_t[x_{t+1} V(\Theta_{t+1}((1-\delta)k_t + i_t))].$$

The first order conditions are given by

$$\begin{aligned} n_t: \quad & w_t = (1-\alpha)a_t k_t^{\alpha} n_t^{-\alpha}, \\ i_t: \quad & 1 = E_t[x_{t+1}V_{k,t+1}\Theta_{t+1}]. \end{aligned}$$

The envelope condition implies

$$\begin{aligned} k_t: \quad V_{k,t} &= \alpha a_t k_t^{\alpha-1} n_t^{1-\alpha} + E_t[x_{t+1}V_{k,t+1}\Theta_{t+1}(1-\delta)], \\ &= \alpha a_t k_t^{\alpha-1} n_t^{1-\alpha} + 1 - \delta, \end{aligned}$$

after imposing $E_t[x_{t+1}V_{k,t+1}\Theta_{t+1}] = 1$. Combining this result with the first order condition for investment yields

$$1 = E_t[x_{t+1}\Theta_{t+1}(r_{t+1}^k + 1 - \delta)], \tag{9}$$

$$r_t^k = \alpha y_t / k_t. \tag{10}$$

Competitive Equilibrium The aggregate resource constraint is given by

$$c_t + i_t = y_t. \tag{11}$$

The equilibrium consists of infinite sequences of quantities $\{k_t, c_t, n_t, y_t, i_t, u_t, J_t, z_t\}_{t=0}^{\infty}$, prices $\{w_t, r_t^k\}_{t=0}^{\infty}$, and exogenous variables $\{\Theta_t, a_t, e_t, \sigma_{a,t}, \sigma_{e,t}\}_{t=0}^{\infty}$ that satisfy (1)-(11) and the exogenous processes, given an state of the economy $\{k_{-1}, a_{-1}, \sigma_{a,-1}, e_{-1}, \sigma_{e,-1}\}$ and the sequences of shocks $\{\varepsilon_{a,t}, \varepsilon_{av,t}, \varepsilon_{e,t}, \varepsilon_{ev,t}\}_{t=0}^{\infty}$.

B SOLUTION METHOD

The equilibrium system of the DSGE model is summarized by $E[g(\mathbf{x}_{t+1}, \mathbf{x}_t, \varepsilon_{t+1})|\mathbf{z}_t, \vartheta] = 0$, where g is a vector-valued function, $\mathbf{x}_t$ is the vector of model variables, ε_t is the vector of shocks, $\mathbf{z}_t$ is the vector of states, and ϑ is the vector of parameters. We discretize the level shocks and volatility processes using the Markov chain in Rouwenhorst (1995), which Kopecky and Suen (2010) show outperforms other methods for approximating autoregressive processes. In our model with a level shock to disaster risk and level and volatility shocks to technology, the bounds on a_t are set to $\pm 5\%$ of the deterministic steady state, while k_t ranges from -60% to 10% of the deterministic steady state to account for disasters. These bounds ensure that simulations contain at least 99% of the ergodic distribution. We specify 9 states for e_t, $\sigma_{a,t}$, and $\varepsilon_{a,t+1}$, and discretize a_t and k_t into 9 and 11 evenly-spaced points, respectively. The product of the points in each dimension, D, is the total number of nodes in the state space ($D = 8{,}019$). The realization of $\mathbf{z}_t$ on node d is denoted $\mathbf{z}_t(d)$. The Rouwenhorst method provides integration nodes, $[\varepsilon_{e,t+1}(m), \sigma_{a,t+1}(m), \varepsilon_{a,t+1}(m)]$, with weights, $\phi(m)$, for $m \in \{1,\ldots,M\}$, where $M = 729$ given that there are three shocks, each with 9 states. The setup in the 2-shock models is analogous.

The vector of policy functions and the realization on node d are denoted by $\mathbf{pf}_t$ and $\mathbf{pf}_t(d)$, where $\mathbf{pf}_t \equiv [n(\mathbf{z}_t), J(\mathbf{z}_t)]$. The following steps outline our policy function iteration algorithm:

1. Use Sims's (2002) `gensys` algorithm to solve the log-linear model. Then map the solution for the policy functions to the discretized state space. This provides an initial conjecture.
2. On iteration $j \in \{1,2,\ldots\}$ and each node $d \in \{1,\ldots,D\}$, use Chris Sims' `csolve` to find the $\mathbf{pf}_t(d)$ that satisfies $E[g(\cdot)|\mathbf{z}_t(d),\vartheta] \approx 0$. Guess $\mathbf{pf}_t(d) = \mathbf{pf}_{j-1}(d)$. Then
   (a) Solve for all variables dated at time t, given $\mathbf{pf}_t(d)$ and $\mathbf{z}_t(d)$.
   (b) Linearly interpolate the policy functions, $\mathbf{pf}_{j-1}$, at the updated state variables, $\mathbf{z}_{t+1}(m)$, to obtain $\mathbf{pf}_{t+1}(m)$ on every integration node, $m \in \{1,\ldots,M\}$.
   (c) Given $\{\mathbf{pf}_{t+1}(m)\}_{m=1}^{M}$, solve for the other elements of $\mathbf{s}_{t+1}(m)$ and compute
   $$E[g(\mathbf{x}_{t+1},\mathbf{x}_t(d),\varepsilon_{t+1})|\mathbf{z}_t(d),\vartheta] \approx \textstyle\sum_{m=1}^{M} \phi(m) g(\mathbf{x}_{t+1}(m),\mathbf{x}_t(d),\varepsilon_{t+1}(m)).$$

   When `csolve` has converged, set $\mathbf{pf}_j(d) = \mathbf{pf}_t(d)$.
3. Repeat step 2 until $\mathrm{maxdist}_j < 10^{-6}$, where $\mathrm{maxdist}_j \equiv \max\{|(\mathbf{pf}_j - \mathbf{pf}_{j-1})/\mathbf{pf}_{j-1}|\}$. When that criterion is satisfied, the algorithm has converged to an approximate solution.

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