

Federal Reserve Bank of Dallas
Globalization and Monetary Policy Institute
Working Paper No. 177

<http://www.dallasfed.org/assets/documents/institute/wpapers/2014/0177.pdf>

Error Correction Dynamics of House Prices: An Equilibrium Benchmark*

Charles Ka Yui Leung
City University of Hong Kong

May 2014

Abstract

Central to recent debates on the "mis-pricing" in the housing market and the proactive policy of central bank is the determination of the "fundamental house price." This paper builds a dynamic stochastic general equilibrium (DSGE) model that produces reduced-form dynamics that are consistent with the error-correction models proposed by Malpezzi (1999) and Capozza et al (2004). The dynamics of equilibrium house prices are tied to the dynamics of the house-price-to-income ratio. This paper also shows that house prices and incomes should be co-integrated, and hence provides a justification of using co-integration tests to detect possible "mis-pricing" in the housing market.

JEL codes: E30, O40, R30

* Charles Ka Yui Leung, Department of Economics and Finance, City University of Hong Kong, Kowloon Tong, Hong Kong. 852-34429604. kycleung@cityu.edu.hk. The author is very grateful to the many comments and suggestions from Nan-Kuang Chen and Fred Kwan. The author also wants to thank (alphabetical order) Kuang-Liang Chang, Ippei Fujiwara, Yifan Gong, Cheng Hsiao, Ronald Jones, Vikas Kakkar, Masanori Kashiwagi, Yuichiro Kawaguchi, Jennifer Lai, Yiting Li, Dan McMillen, Gulseren Mutlu, Alexander Powell, Isabel Yan, Eden Yu, seminars participants of different institutions and conferences, and especially an anonymous referee and Thomas Davidoff for comments and suggestions, as well as the City University of Hong Kong for financial support. The work described in this paper was partially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China [Project No. CityU 144709]. Edward Tang provides excellent research assistance. The views in this paper are those of the author and do not necessarily reflect the views of the Federal Reserve Bank of Dallas or the Federal Reserve System.

1 Introduction

This paper has several aims. First, it contributes to an emerging concern on relating economic fundamentals to asset prices. For instance, many authors have discussed whether the housing boom that occurred before the 2008 crisis was due to “over-pricing”.¹ Obviously, the level of the “*fundamental house price*” (FHP) needs to be determined before the degree of “over-pricing” or “under-pricing” can be found. The potential issue of “mis-pricing” in the housing market is also related to the debate on the role of the central bank. Some authors argue that the central banks should act proactively once the asset prices (both of stocks and of housing) deviate significantly from the levels considered to be consistent with economic fundamentals.² Even if everyone agreed to institute a “proactive” central bank, there remains a gap to be filled through identifying “*econometrically implementable*” tests that *define and detect* “*significant deviations*” in asset prices. In other words, an empirical determination of the FHP is central to both areas of debate. This paper extends the literature by providing a simple theory of house price dynamics when the housing market and the macroeconomy are driven solely by economic fundamentals. This theory can then be used as a benchmark for detecting the “deviations” that are potentially “mis-pricing”.

Similar attempts have been made previously. For instance, Malpezzi (1999) relates the movement of house-price-to-income ratio to the house price dynamics and makes two

¹The literature is too large to be reviewed here. Among others, see Sowell (2009) for a review of the literature and related issues.

²For instance, the Economist magazine (2011) summarizes that, “Since the financial crisis in 2007 central banks have expanded their remit, either at their own initiative or at governments’ behest, well beyond conventional monetary policy. They have not only extended the usual limits of monetary policy by buying government bonds and other assets... They are also taking on more responsibility for the supervision of banks and the stability of financial systems.” See also Taylor (2009) for a related discussion.

Clearly, there are alternative views on the related issues and the discussion is still on-going.

conjectures. First, the house-price-to-income ratio is a constant in the long run (this conjecture is hereafter denoted by $M1$). Second, house price changes do not directly depend on price lag, but instead on the house-price-to-income ratio (in both the current period and previous periods) in a format that exhibits certain features of an error-correction model (this conjecture is hereafter denoted by $M2$). Malpezzi's paper clearly addresses the concerns of the general public in addition to those of official agents, as the house-price-to-income ratio is often used as a measure of whether house prices have deviated from "fundamental" prices. For instance, the *Wall Street Journal* (2011) comments that "...For decades, price-to-income levels have moved in tandem, with a specific housing market's prices rising or falling in line with local residents' incomes. Many economists say that makes the price-to-income ratio a good gauge for determining whether housing is undervalued or overvalued for a given market." Regardless of whether all economists would agree with this statement, it reflects the situation as perceived by the media. In a research note of the Parliament of the United Kingdom, Keep (2012) comments that "*The ratio of house prices to income is a key indicator of the relative affordability of owner-occupation.*" A more systematic study of the relationship between house-price-to-income ratio and movements in house price may thus be of value.

In contrast, Capozza et al. (2004) assert that house prices follow a second-order difference equation *without income* explicitly appearing in the equation (this conjecture is hereafter denoted by $C1$). Their simple and elegant model finds support from a dataset with 62 metropolitan statistical areas (MSAs) in the United States from 1979 to 1995 (*hereafter*). More recently, Glindro et al. (2011) find support for $C1$ from nine Asia-

Pacific countries.³ Thus, some *common patterns* for house price dynamics seem to exist across countries.

Note also that while Malpezzi (1999) includes the house-price-to-income ratio in the empirical model, the model of Capozza et al (2004) only contains house price information. These empirical models may thus represent different degrees of direct dependence of house prices on income. There are, of course, other possible forms of error correction models for house prices. For reasons of space, however, this paper focuses on building a simple model to relate directly to these empirical models. More specifically, this paper addresses the following questions arising from the two papers.

1. Malpezzi (1999) and Capozza et al. (2004), among others, provide empirical models for the dynamics of house prices while leaving the theoretical side open. Is there a way to “rationalize” these empirical models of house price dynamics in an equilibrium setting with solid micro-foundation?
2. While these papers are innovative and insightful, the “error-correction structures” in their models deviate from that in conventional error correction models (ECMs). However, these models have achieved empirical successes. Can we provide a theoretical justification for their abstraction of the dynamics of income in models of house price?
3. Both Malpezzi (1999) and Capozza et al (2004) use MSA data from the U.S. with almost identical sampling periods. Is it possible that there are some links between

³The nine countries are Australia, China, Hong Kong, Korea, Malaysia, New Zealand, Philippines, Singapore and Thailand.

the two models?

This paper takes the first step to address these questions. In particular, this paper attempts to build a dynamic stochastic general equilibrium (DSGE) model in which both house prices and aggregate output are derived endogenously. The house price dynamics derived from this model can take a form that is similar either to (1) or to (2). In other words, we can reconcile (M1, M2, C1) in an *unifying framework*. Thus, this paper also establishes a strong link between the econometrics literature and the DSGE literature.⁴ It may be of interest independently because while error-correction models are often interpreted as evidence of “disequilibrium”, in this paper we derive an error-correction model for house prices in an equilibrium setting.⁵

This paper also contributes to the recent macro-housing literature. Providing a comprehensive survey of this emerging literature is beyond the scope of this paper;⁶ instead, a few contributions are highlighted. Greenwood and Hercowitz (1991) provides one of the earliest studies of the different allocation of business and household capital in a dynamic, general equilibrium setting. Jin and Zeng (2004) and Ortalo-Magne and Rady (2006), among others, place emphasize on collateral constraints and how the endogeneity of the house price provides feedback into the macroeconomy. Iacoviello (2005), Iacoviello and Neri (2010), and Iacoviello and Pavan (2012), among others, concentrate on the quantita-

⁴In some earlier literature, such as Kydland and Prescott (1996), DSGE is sometimes interpreted as a substitute of econometrics. In the more recent literature, such as those surveyed by Fernandez-Villaverda (2010), DSGE are often estimated with sophisticated econometrics techniques.

⁵It is well known that the reduced form dynamics of the DSGE model can be summarized by a VAR structure and co-integration test can be performed, as demonstrated by King et al, 1991; King, Plosser and Rebelo, 2002, among others. To our knowledge, theoretically derive an error-correction structure and examine the cointegration relationship among variables in a DSGE model is rare, and this paper takes a preliminary step towards this direction.

⁶Among others, see Leung (2004), Van Nieuwerburgh (2012), for a review of the literature.

tive aspect and include calibration to match different aspects of the U.S. housing market and the macroeconomy. Ried and Uhlig (2009) build a two-sector DSGE model to numerically mimic the house-price-to-GDP ratio and the stock-price-to-GDP ratio. Chen et al (2012) study how house prices and mortgage premiums interact in a DSGE framework. Clearly, it is important to know how far the observed variations in house prices can be attributed to different effect and to design economic policies accordingly. However, such models are typically technically involved and non-specialists may not be able to fully understand the underlying mechanisms. On the applied theoretical front, Jin and Zeng (2007) focus on the implications of different policies, given the collateral constraints. This paper complements the efforts of these researchers by simplifying a standard DSGE model and deriving the *econometric implications* of that model. In particular, it achieves several goals: (a) it shows how house prices would evolve when driven purely by the aggregate productivity shock, (b) it studies the stationarity of both income and house prices under a general equilibrium setting, and (c) it theoretically studies the co-integration of the two variables. These theoretical implications are clearly *testable* and hence provides a benchmark for future work on both the empirical and the theoretical fronts.

The organization of this paper is straightforward. The following sections briefly review the work of Malpezzi (1999) and Capozza et al (2004), and present the model and the results. The final section provide some concluding remarks and all the proofs are presented in the appendix.

2 The Model

Before introducing the model formally, the formulation and results of Malpezzi (1999) and Capozza et al (2004) are briefly reviewed. Malpezzi asserts that

$$dP_t = \beta_0 + \beta_1 \left(\frac{P_{t-1}}{Y_{t-1}} - k \right) + \dots + \beta_n \left(\frac{P_{t-n}}{Y_{t-n}} - k \right) + \gamma_1 \left(\frac{P_{t-1}}{Y_{t-1}} - k \right)^3 + \dots + \gamma_n \left(\frac{P_{t-n}}{Y_{t-n}} - k \right)^3 + X\alpha + \varepsilon_t, \quad (1)$$

where P_t is the house price, Y_t is the income, k is some long-run house-price-to-income ratio, X is a vector of control variables, and ε_t is the error term. According to (1), if the house-price-to-income ratio is a constant $\frac{P_t}{Y_t} = k$ for all period t , then the change of house price dP_t will be driven merely by the control variables X . He finds empirical evidence from 133 MSAs from 1979 to 1996. Follow-up research mainly focus on M1 and the evidence has been somewhat mixed.⁷

Capozza et al (2004) claim that house price dynamics can be summarized by the following equation:

$$dp_t = \alpha_1 dp_{t-1} + \alpha_2 (p_{t-1}^* - p_{t-1}) + \alpha_3 dp_t^*,$$

$$\text{or } p_t + \alpha'_1 p_{t-1} + \alpha'_2 p_{t-2} = \alpha'_3 p_t^* + \alpha'_4 p_{t-1}^*, \quad (2)$$

where p_t is the log house price, p_t^* is some long run equilibrium house price given the period t “*exogenous explanatory variables*”. Capozza et al (2004) recognize that *in general*, the right hand of (2) is *stochastic*. For convenience, they assume that p_t^* to be a constant for all t .⁸

⁷It is beyond the scope of this paper to review that literature. Among others, see Gallin (2006), Klyuev (2008), among others.

⁸Notice that when the whole series $\{p_t^*\}$ is assumed to be constant over time, it is as if suggesting

Interestingly, while both models involve an aspect of “error correction” in the data, they significantly *deviate from the conventional ECMs* which has been widely used. According to Engle and Granger (1987, p.251-2), “If x_t is a vector of economic variables, then they may be said to be in equilibrium when the specific linear constraint $\alpha'x_t = 0$ occurs. In most time periods, x_t will not be in equilibrium and the univariate quantity $z_t = \alpha'x_t$ may be called the equilibrium error. If the equilibrium concept is to have any relevance for the specification of econometric models, the economy should appear to prefer a small value of z_t rather than a large value.... a class of models, known as *error-correcting*, allows long-run components of variables to obey equilibrium constraints while short-run components have a flexible dynamic specification. A condition for this to be true, called *co-integration*, was introduced by Granger (1981) and ...” (Italics added). Thus, if one is going to build an ECM following Engle and Granger (1987), it may be natural to conjecture the following form,

$$\begin{pmatrix} \Delta y_t \\ \Delta p_t \end{pmatrix} = A_0 + B_1 \begin{pmatrix} y_{t-1} \\ p_{t-1} \end{pmatrix} + \sum_{j=1}^n A_j \begin{pmatrix} \Delta y_{t-j} \\ \Delta p_{t-j} \end{pmatrix} + \vec{\varepsilon}_t, \quad (3)$$

where B_1 , A_j , $j = 0, 1, \dots$ are matrices with the appropriate dimensions. In light of (3), the income dynamics is simply taken as given in Malpezzi (1999), and is completely missing in Capozza et al. (2004). However, it is clear that the models of both Malpezzi (1999) and Capozza et al. (2004) achieve empirical successes that apparently justify their formulations. We therefore consider it important to consider a theoretical model which would help us to understand *why* their formulations work; this is the subject of the next

that the factors which determine the house price have been reflected in the previous period house prices and other adjustment parameters, and explicit dependence on other variables are un-necessary. In a way, it somehow shares the spirit of “efficient market hypothesis.” It may not be the intention of the original authors, yet some subsequent authors do interpret the model this way.

section.

2.1 A Simple, Dynamic Stochastic General Equilibrium (DSGE) Model

Our model builds on Greenwood and Hercowitz (1991) and Kan et al. (2004), and an informal overview may be helpful. This is a discrete time model with an infinite horizon. The economy is populated by a continuum of infinite-lived agents. The population is fixed and is normalized to unity. There are three goods: a non-storable consumption good, residential property and business capital. While the consumption good is perishable, the other two goods are durable, where the depreciation rate of business capital is commonly recognized to be higher than that of residential property.⁹

The economy is subject to persistent productivity shocks that affect production, consumption and investment opportunities. Forward-looking agents would therefore choose the consumption of non-durable goods and durable housing, based on their budgets and expectations of the future. Since the short-run supply of housing is relatively inelastic, house prices would change according to the fluctuations in demand. An investment in residential property would respond to changes in output and house prices. Thus, a positive productivity shock today would be associated with increases in output and house prices. These increases would in turn encourage more investment in residential property, leading to an increased supply of housing in the future and hence exerting a downward pressure on future house prices. Thus, house prices would exhibit a form of “mean reversion” or “error-correction” dynamics, even when all economic agents are rational and forward-looking.

⁹The word “residential property”, “housing” and “residential capital” will be used interchangeably in this paper.

More formally, in each period t , $t = 1, 2, \dots$, the representative agent maximizes the expected value of the lifetime utility $E_t \sum_{m=t}^{\infty} \beta^{m-t} u(C_m, H_m, N_m)$, which is the discounted sum of the periodic utility $u(C_m, H_m, N_m)$, where β is the discounted factor, $0 < \beta < 1$, C_m is the amount of consumption in period m , H_m is the stock of housing (or, residential property; these terms are used interchangeably throughout this paper) owned by the representative agent in period m , and N_m is the number of working hours provided by the agent, $0 < N_m$, $m = t, (t+1), \dots$. Following the literature, we assume that the preference of the representative agent is separable. Formally, the utility function takes the following form,

$$u(C_t, H_t, N_t) = \ln C_t + w_H \ln H_t - w_L \frac{(N_t)^{1+\eta}}{1+\eta}, \quad (4)$$

where both w_H and w_L are positive preference parameters determining consumers' resource allocation among consumption, residential property and leisure. The assumed positivity of η ensures that the labor supply curve is upward sloping.

The production side is relatively simple. The goods production technology is represented by a Cobb-Douglas function,

$$Y_t = A_t (K_t)^\alpha (N_t)^{1-\alpha}, \quad (5)$$

where $0 < \alpha < 1$, and exhibits constant return to scale in business capital and labor. Throughout this paper, the terms “output” and “income” will be used interchangeably for Y_t . The amount of output, however, depends not only on the amount of business capital and labor, but also on productivity A_t , which fluctuates over time. In the macroeconomic literature, it is commonly assumed that the log of productivity follows an AR(1) process,

$$a_t = \rho a_{t-1} + \varepsilon_t, \quad (6)$$

where $a_t = \ln(A_t)$, $0 < \rho \leq 1$, and ε_t is a white noise process.¹⁰

The evolution of the different kinds of capital stock in the model is also simple and standard. For tractability, this paper follows Hercowitz and Sampson (1991) and Kan et al. (2004), in their assumption of the laws of motion of business capital K_t and residential property H_t ,

$$K_{t+1} = B_K (K_t)^{1-\delta} (I_t^K)^\delta, \quad (7)$$

$$H_{t+1} = B_H (H_t)^{1-\gamma} (I_t^H)^\gamma + H_t^S, \quad (8)$$

where $0 < B_K, B_H < \infty$, $0 < \delta, \gamma < 1$, and H_t^S represents the amount of existing housing stock purchased from the secondary market.¹¹ In this context, it may be instructive to consider δ and γ as the “depreciation factors” of the business capital and residential property respectively. Notice that in (7) and (8), the investment terms I_t^K and I_t^H represents not only the new investment, but also the resources devoted to the completion of existing projects and the maintenance and renovation of existing capital assets and residential properties, and hence can be very significant amounts.

The dynamic optimization problem of the representative agent can be summarized in a Bellman equation,

$$V(H_t, K_t) = \max u(C_t, H_t, N_t) + \beta E_t V(H_{t+1}, K_{t+1}),$$

subject to the budget constraint

$$C_t + P_t H_t^s + I_t^K + I_t^H \leq Y_t, \quad (9)$$

¹⁰By that we mean $E_t(\varepsilon_{t+k})$ is independent of k , $cov(\varepsilon_t, \varepsilon_{t+q}) = 0$, $\forall q > 0$, and $var(\varepsilon_t) = \sigma_\varepsilon^2$ which is a constant.

¹¹Lau (2002) shows that these laws of motion is equal to the conventional counterpart, $K_{t+1} = \rho K_t + I_t^K$, when the DSGE model is log-linearly approximated.

and the laws of motion for capital and housing, (7) and (8), where $u(C_t, H_t, N_t)$ is given by (4) and Y_t is given by (5). Implicitly, it is assumed that the representative agent can buy or sell existing housing stock at unit price P_t .

To solve this rather complicated system of equations, equilibrium conditions need to be imposed. Following Lucas (1978), we impose the natural restriction that the net sale of residential properties among agents is zero in equilibrium,

$$H_t^S = 0, \forall t. \tag{10}$$

Equipped with all these conditions, we are now ready to define the stationary equilibrium of the model.

Definition 1 *For a given sequence of productivity shocks $\{A_t\}_{t=0}^{\infty}$, a stationary equilibrium is a sequence of quantities $\{C_t, K_t, H_t, I_t^K, I_t^H, N_t\}_{t=0}^{\infty}$, and a sequence of housing prices $\{P_t\}_{t=0}^{\infty}$, such that the representative agent maximizes his expected life-time utility, subject to the constraints (7), (8), and (9), Meanwhile, the housing market clears, implying equation (10) holds.*

In this model, the equilibrium can be solved explicitly. The solution strategy is simple. The quantities are solved first, and are then followed by prices. All proofs are presented in the appendix. We first show that the equilibrium quantities can be characterized by the following equations.

Proposition 1 *Under some conditions, the evolution of the equilibrium quantities can be*

characterized by these following equations:

$$\begin{aligned}
C_t &= \phi_c Y_t, \\
I_t^K &= \phi_K Y_t, \\
I_t^H &= \phi_H Y_t, \\
N_t &= \eta_N,
\end{aligned} \tag{11}$$

for some constant ϕ_c, ϕ_K, ϕ_H and η_N , such that $0 < \phi_c, \phi_K, \phi_H < 1, 0 < \eta_N$.

A few comments may improve understanding of the intuitions behind the results. Intuitively, if there is a good productivity shock, then output Y_t is expected to increase. The income effect encourages the economic agents to increase the consumption-to-output ratio. On the other hand, the substitution effect suggests the economic agent to invest more because with more output, the marginal cost of investment decreases. Since our utility function is in log form, the income effect and substitution effect exactly offset each other. And with some technical conditions that are detailed in the appendix, the consumption-to-output ratio and different investment-to-output ratios in the model remain constant. In addition, the utility from is separable, and hence the labor supply depends on the product of the marginal utility of consumption and the output, (λ_{1t}, Y_t) . Intuitively, if there is a good productivity shock, then output Y_t is expected to increase. However, with a higher income, consumption will also increase, so that marginal utility of consumption λ_{1t} will decrease. And with our separable log utility functional form, the two effects exactly offset each other and hence the labor supply becomes constant.

Using (11), we can further characterize the entire dynamical system. First, we take the natural logarithm for different variables. For instance, $c_t = \ln C_t, y_t = \ln Y_t$, etc. This

procedure not only facilitates the subsequent analysis, but is also the procedures used in many empirical works and hence facilitates communication between the theoretical and empirical work in this area. Notice further that most control variables depend on the output Y_t , which in turns depends on the productivity shock, capital stock and labor (see equation (5)). Following Sargent (1979), it is in fact a *block-recursive system*, meaning that once we solve the dynamics of the “core” sub-system, the entire dynamical system can be easily solved. Formally, it means that the model economy is described by the following equation system,

$$\begin{aligned}
y_t &= a_t + \alpha k_t + b_y, \\
k_{t+1} &= b_k + (1 - \delta) k_t + \delta i_t^k, \\
h_{t+1} &= b_h + (1 - \gamma) h_t + \gamma i_t^h, \\
i_t^k &= b_{ik} + y_t, \\
i_t^h &= b_{ih} + y_t,
\end{aligned} \tag{12}$$

where $b_y, b_k, b_h, b_{ik}, b_{ih}$, are all constant. Note further that (12) is driven by a “core” sub-system, which is the vector \vec{y}_t , where

$$\vec{y}_t = \begin{pmatrix} y_t \\ k_t \end{pmatrix}.$$

The following result describes the dynamics of \vec{y}_t .

Proposition 2 *The dynamics of \vec{y}_t is dictated by the following vector equation,*

$$\vec{y}_{t+1} = M_0 + M_1 \vec{y}_t + \vec{u}_{t+1}, \tag{13}$$

where M_0, M_1 are matrices of constant, \vec{u}_t is a vector of shock terms, $\vec{u}_t = \begin{pmatrix} a_{t+1} \\ 0 \end{pmatrix}$. In fact, we can express, in log form, the output, physical capital stock, and residential housing

stock as a summation of previous period productivity shocks,

$$\begin{aligned}
y_t &= b_y'' + a_t + (\alpha\delta) \sum_{i=0}^{\infty} (1 - \delta + \alpha\delta)^i a_{t-1-i}, \\
k_t &= b_k'' + \delta \sum_{i=0}^{\infty} (1 - \delta + \alpha\delta)^i a_{t-1-i}, \\
h_t &= b_h'' + \gamma a_{t-1} + \gamma \sum_{i=0}^{\infty} \delta_h(i) a_{t-2-i},
\end{aligned} \tag{14}$$

where b_x'' , $x = y, k, h$ are constant, $\delta_h(i)$ is a function of parameters and i .

In a sense, this result is not very surprising. For instance, Slutsky (1937) shows that a cyclical stochastic process can be decomposed as a summation of uncorrelated random variables. Thus, (14) can be interpreted as a *re-statement* of Slutsky's result in a DSGE context. A consequence of (14) is that the stochastic properties of the economic variables (the output y_t , physical capital stock k_t , and residential housing stock h_t) will depend on the stochastic properties of the productivity shock a_t . Following Cooley (1995), we assume that the productivity shock follows an AR(1) process, as reflected by (6). Empirically, the productivity shock process is not only *serially correlated*, but is also very close to unit root (Cooley, 1995). According to some authors, the persistence parameter, ρ , is sometimes statistically indistinguishable from unity. We will therefore study both the case of “mean-reverting” (or “temporary”) case, i.e. where $0 < \rho < 1$, and the case of “unit root” (or “permanent”) case, i.e. where $\rho = 1$. For simplification, we will focus on the case of mean-reverting productivity shock in the text and present the case of permanent shock in the appendix.

Proposition 3 *The stationarity of y_t depends on the persistency of the productivity shock a_t , which is defined in (6). In particular, when we express y_t in the form of $\sum_{j=0}^{\infty} \psi_j^{(y)} \varepsilon_{t-j}$,*

and if $0 < \rho < 1$, i.e. the productivity shock a_t is a temporary shock, then the income y_t is stationary,

$$\sum_{j=0}^{\infty} |\psi_j^{(y)}|^2 < \infty. \quad (15)$$

Notice that in general, $\{\psi_j^{(y)}\}$ would depend on the value of ρ . The appendix shows that if temporary productivity shocks are permanent instead of temporary, the output will be non-stationary. As we show below, such distinctions will also appear in the housing market. Our next step is to understand the house price dynamics within this model. We first relate the macro-dynamics to the housing market dynamics in the model. As we have characterized the evolution of the major quantities in the model, namely, the output, physical capital stock and residential property stock, as a summation of shocks in different periods, the following proposition will help us to establish a dynamic relationship between the house price (in real terms) and the other macroeconomic variables.

Proposition 4 *In log form, the dynamics of the house price can be related to the dynamics of output and the housing stock in the following way:*

$$\begin{aligned} p_t &= b_p + y_t - h_{t+1}, \\ &= b'_p + \gamma \sum_{i=1}^{\infty} (1 - \gamma)^i (y_t - y_{t-i}), \end{aligned} \quad (16)$$

where b_p, b'_p are constant.

The interpretation of (16) is clear. The first expression indicates that the house price depends positively on the level of current level aggregate output and negatively on the

foreseen amount of residential property stock. Thus, other things being equal, an anticipation of an increase in the future stock of housing will depress the house price today. The house price is “*forward-looking*” in this sense. The second expression requires more explanation. Notice that, by definition, the term $(y_t - y_{t-i}) = \ln Y_t - \ln Y_{t-i} = \ln(Y_t/Y_{t-i})$, which is the economic growth between period t and $(t - i)$. Thus, the second expression indicates that the house price (in real terms) can be interpreted as a *weighted sum of the economic growth* between the current period and different previous periods. Higher economic growth leads to a higher house price. Notice that the weight allocated to the economic growth between period t and $(t - i)$ is $(1 - \gamma)^i$. As $0 < (1 - \gamma) < 1$, the value of the term $(1 - \gamma)^i$ decreases with i , such that periods of more distant past (i.e. higher values of i) will carry less weight. In other words, if the output level at some particular time t' is unusually high, it would lead to a higher level of house price $p_{t'}$. The impact, however, will die off over time. It is because if $y_{t'}$ is unusually high, then for $t > t'$, it is likely that $(y_t - y_{t'})$ will become small and hence will constrain further increase in house price. The model thus implies that *continuous increase in house price can only be sustained by persistent economic growth*.

The next result concerns the stationarity of p_t . This is the housing market counterpart of (15). In other words, there are some similarities between the aggregate output and the house market price in terms of the stationarity.

Proposition 5 *The stationarity of p_t depends on the persistency of the productivity shock a_t , which is defined in (6). In particular, when we express p_t in the form of $\sum_{j=0}^{\infty} \psi_j^{(p)} \varepsilon_{t-j}$, and if $0 < \rho < 1$, i.e. the productivity shock a_t is mean-reverting, then the house price p_t*

is stationary,

$$\sum_{j=0}^{\infty} \left| \psi_j^{(p)} \right|^2 < \infty. \quad (17)$$

Equipped with the stationarity results of the output and the (real) house price, we can now present our main theorem, which relates the dynamics of the house price growth and the dynamics of output.

2.2 Main Result

The following theorem characterizes the growth rate of the house price as a function of house-price-to-income ratio (or, equivalently, house-price-to-output ratio) in the current and previous periods.

Theorem 1 *The house price growth rate and the house price-to-output ratio can be related in the following way:*

$$dp_{t+1} = \frac{(1-\gamma)}{\gamma} \{ [\varkappa_{t+1} - \varkappa_t] - [\varkappa_t - \varkappa_{t-1}] \}, \quad (18)$$

where dp_{t+1} is the house price growth factor,

$$dp_{t+1} \equiv (p_{t+1} - p_t) = \ln(P_{t+1}/P_t), \quad (19)$$

and \varkappa_t is the log of the house price-to-output ratio,

$$\varkappa_t \equiv y_t - p_t = \ln(Y_t/P_t). \quad (20)$$

The significance of this theoretical result needs to be put into context. In the media and many applied studies, the house-price-to-income ratio is usually used to measure

of whether a “housing bubble” exists. Yet formal studies on the relationship between the house price dynamics and the house-price-to-income ratio are rare. This theoretical result attempts to fill this gap. Notice that in (18), all variables are all in logarithmic form. Thus, the variable $\varkappa_t \equiv y_t - p_t = \ln(Y_t/P_t) = \ln(P_t/Y_t)^{-1} = -(p_t - y_t)$, which is the negative of the house-price-to-income ratio. Hence, $[\varkappa_{t+1} - \varkappa_t]$ is a measure of the change in the house-price-to-income ratio. Consequently, the right hand side term $\{[\varkappa_{t+1} - \varkappa_t] - [\varkappa_t - \varkappa_{t-1}]\}$ is a measure of the change of the growth rate of the house-price-to-income ratio. Alternatively, if we may draw an analogy from physics, the term $\{[\varkappa_{t+1} - \varkappa_t] - [\varkappa_t - \varkappa_{t-1}]\}$ is a measure of the *acceleration* of the house-price-to-income ratio. The left hand side term $dp_{t+1} \equiv (p_{t+1} - p_t) = \ln(P_{t+1}/P_t)$ is simply the (gross) rate of growth in house prices. Thus, this theoretical result states that the growth rate of the house price depends on the acceleration of the house-price-to-income ratio. Thus, the media attention to focus on the *level* or the *rate of change* of the house-price-to-income ratio could be misplaced.

This theoretical result not only sheds light on the dynamics of the house-price-to-income ratio but also helps us to understand the stylized facts mentioned at the beginning of this paper. Formal analysis on these is now in order.

2.3 Relationship with Capozza et al (2004)

In this subsection, we show that a re-statement of our theoretical result (equation (18)) sheds light on stylized fact C1. Recall that Capozza et al. (2004) assert that the house

prices follow a second-order equation, (2),

$$dp_t = \alpha_1 dp_{t-1} + \alpha_2 (p_{t-1}^* - p_{t-1}) + \alpha_3 dp_t^*,$$

$$\text{or } p_t + \alpha'_1 p_{t-1} + \alpha'_2 p_{t-2} = \alpha'_3 p_t^* + \alpha'_4 p_{t-1}^*,$$

where α_i , $i = 1, 2, 3$, are parameters to be estimated, dp_t is defined in (19), and p_t^* is some long-run equilibrium house price given the period t “*exogenous explanatory variables*”. Capozza et al. (2004) recognize that, *in general*, the right-hand side is *stochastic*. For convenience, they assume that p_t^* to be a constant for all t . Interestingly, they find empirical support for their formulation. The next proposition shows that we can also re-write (18) in a *comparable* manner.

Proposition 6 *The house price dynamics in (18) can be expressed as a second order difference equation in the following form,*

$$p_t + \gamma'_1 p_{t-1} + \gamma'_2 p_{t-2} = (1 - \gamma) [(dy_t - dy_{t-1})], \quad (21)$$

where γ'_1, γ'_2 are parameters, and dy_t is the change in log-income,

$$dy_t \equiv y_t - y_{t-1}. \quad (22)$$

Notice that by definition, $dy_t \equiv y_t - y_{t-1} = \ln(Y_t/Y_{t-1})$, the log of the gross growth rate. If the economic growth rate g_{yt} is small, where $g_{yt} \equiv (Y_t/Y_{t-1}) - 1$, then $dy_t \approx g_{yt}$. Thus, the right-hand side of (21) is approximately equal to a fraction of the *change in economic growth rate*.¹² When the economic growth rate is relatively stable, which may be the case for the sampling period of Capozza et al. (2004) (from 1979 to 1995), the

¹²In the appendix, we provide more analysis on this.

right-hand side of (21) can indeed be approximated by a constant. In other words, the assumption made in Capozza et al. (2004) may be justifiable not only on the empirical ground, but also on theoretical grounds when the economic growth rate does not vary too much. Therefore, the dynamic model proposed here provides *an equilibrium interpretation* of the empirical results in Capozza et al. (2004).

2.4 Relationship with Malpezzi (1999)

Our theoretical result expressed in equation (18) can also help us to understand M1 and M2. Recall that Malpezzi (1999) considers the following empirical model, (1),

$$dP_t = \beta_0 + \beta_1 \left(\frac{P_{t-1}}{Y_{t-1}} - k \right) + \dots + \beta_n \left(\frac{P_{t-n}}{Y_{t-n}} - k \right) + \gamma_1 \left(\frac{P_{t-1}}{Y_{t-1}} - k \right)^3 + \dots + \gamma_n \left(\frac{P_{t-n}}{Y_{t-n}} - k \right)^3 + X\alpha + \varepsilon_t.$$

Notice that our theoretical result (equation (18)) is in logarithmic form. If we apply the Taylor expansion on all the log terms of (18), our result will be approximated by the formulation of Malpezzi (equation (1)).¹³ Thus, our theoretical model has successfully derived M2 and the equation (18) can indeed be an equilibrium interpretation of the stylized fact discovered by Malpezzi (1999). It also demonstrates that an error-correction structure can be compatible with a DSGE model.

We now connect equation (18) to the last stylized fact M1, which concerns the steady state of the house price, and the house-price-to-income ratio. The following result is immediate from (18).

Corollary 1 *Based on (18), if the house price-to-income ratio reaches its steady state,*

¹³Obviously, we need to remove the quadratic terms, in order to “fit (18) into (??),” so to speak.

$\varkappa_{t+1} = \varkappa_t, \forall t$, then the house price will also reaches its steady state value, $p_{t+1} = p_t$.

Notice that *if* the house-price-to-income ratio reaches a steady state, by (20), $\varkappa_t \equiv y_t - p_t = \ln(Y_t/P_t) = \varkappa$, where \varkappa is a constant. Thus,

$$Y_t/P_t = \exp(\varkappa).$$

This is exactly the conjecture M1 proposed by Malpezzi (1999)! However, a steady state *may not exist* for non-stationary processes. Thus, to understand the dynamics of the house price, it is necessary to *verify the stationarity* of the house-price-to-income ratio. To verify the stationarity of a stochastic process, it is natural to apply the Wold's decomposition theorem, which requires us express the variable $\varkappa_t \equiv (y_t - p_t)$ as a (potentially) infinite sum of white noise.¹⁴ The following proposition expresses the idea formally,

Proposition 7 *The house price-to-income ratio (in log) can be expressed as a weighted sum of the current period and previous periods of productivity shocks,*

$$(y_t - p_t) = \widehat{b}_{yp} + \gamma a_t + \gamma \sum_{i=0}^{\infty} \delta_h(i) a_{t-1-i}, \quad (23)$$

where \widehat{b}_{yp} is a constant term, and $\{\delta_h(i)\}, \{\delta'_h(i)\}$ are functions of parameters.

Corollary 2 *At the deterministic steady state, $\{a_t\} = 0$, the house price-to-income ratio (in log) is a constant, $(y_t - p_t) = \widehat{b}_{yp}$. In other words, (Y_t/P_t) is a constant.*

The corollary shows that the house-price-to-income ratio becomes a constant at the deterministic steady state, which is consistent with M1 and the empirical finding of Malpezzi

¹⁴See the appendix for more discussion on this.

(1999). In general, however, $\{a_t\} \neq 0$. Moreover, as we have assumed in (6), it is serially correlated, i.e. $cov(a_t, a_{t-k}) \neq 0$ for $k > 0$. Thus, to verify the stationarity of the house-price-to-income ratio $(y_t - p_t)$, we need to express it as an infinite sum of serially uncorrelated disturbance terms, following the Wold's decomposition theorem. The following proposition addresses this.

Proposition 8 *Assume that (6) holds. We can then write the house price-to-income ratio in the following form,*

$$(y_t - p_t) = \hat{b}_{yp} + \sum_{i=0}^{\infty} \delta'_h(i) \varepsilon_{t-i}, \quad (24)$$

where ε_t is a white noise. Furthermore, if $0 < \rho < 1$, i.e. the productivity shock a_t is a temporary shock, the stochastic process of log house price-to-income ratio is stationary,

$$\sum_{j=0}^{\infty} |\delta'_h(j)|^2 < \infty, \quad (25)$$

The intuition of the result is simple. If the productivity shock is temporary ($0 < \rho < 1$), a positive shock today may not have much impact on the discounted lifetime wealth as the representative agent rationally expects a future negative shock to offset the current period's positive shock. As a result, the house price does not increase much and the house price-to-income ratio remains stationary. In the empirical literature, many authors do not consider whether the house-price-to-income ratio is stationary. Instead, they investigate whether house prices and income are co-integrated. Interestingly, our model can also shed light on this issue. Notice that in logarithmic form, the house-price-to-income ratio is

simply $(y_t - p_t)$ and hence the stationarity of the house-price-to-income ratio is equivalent to the co-integration of the two variables.

Corollary 3 *If $0 < \rho < 1$, the house price and income (both in log) are co-integrated.*

In the appendix, we show that if $\rho = 1$, the house-price-to-income ratio is not stationary, i.e. $(y_t - p_t)$ is not stationary. However, the two variables can still be co-integrated if we can find a constant θ such that $(y_t - \theta p_t)$ is stationary. The appendix shows that this is indeed the case. The important lesson here is that regardless of whether the productivity shock is mean-reverting or permanent, house prices and income are co-integrated. Thus, this paper provides a *theoretical foundation* for the empirical practice of applying co-integration tests to the two variables.

3 Concluding Remarks

This paper accomplishes several objectives. It builds a DSGE model that exhibits several desirable features. In a reduced form, the model displays an error-correction structure that is consistent with the empirical work of Malpezzi (1999), Capozza et al. (2004), and others. It also relates the degree of persistence of a productivity shock to the stationarity of the income and house prices, and even of the house-price-to-income ratio. The model suggests that it is possible for some empirical studies to find stationary house-price-to-income ratio while others do not, because the shock processes in *different empirical samples* can process *different stochastic properties* and hence alter the empirical results. This paper also shows that even when the house-price-to-income ratio is not stationary, it is still possible to obtain a co-integration relationship between house prices and income. Thus, it provides a

micro-foundation for the empirical practices in the literature of applying the co-integration test between house prices and income. In addition, as this paper demonstrates that it is possible for a DSGE model to have an error-correction structure, the empirical validity of the error-correction model can be consistent with an equilibrium interpretation of the dynamics of output and house prices.

This paper makes several simplifying assumptions, which naturally become limitations. We will briefly discuss these here and leave it to future studies to improve on these dimensions. First, this paper assumes that there is only a national housing market and only one source of shock that affects both the aggregate output and the housing market. In practice, different regional housing markets may be subject to different shocks and may behave differently.¹⁵ To address the regional heterogeneity, this model needs to be extended to allow for the existence of (ex post) heterogeneous regions and to possibly allow for region-specific shocks to affect different regions. In addition, regional labor markets and, in general, the “frictions” involved in any factor (labor, capital, . . .) to move across regions, or even the financial instruments available for agents to diversify risk, may affect the regional house price dynamics.¹⁶ Clearly, more research is needed into the equilibrium prediction of regional house price dynamics.

As the current paper focuses on the productivity shock as a driving force for both the aggregate output and house prices, it inevitably ignores other channels. For instance, the

¹⁵Among others, see Davidoff (2013) for a review of the literature.

¹⁶Recently, there are some attempts along this line. For instance, Leung and Teo (2011) study a multi-region DSGE with both aggregate and region-specific shocks. They allow agents to hold stock as well as housing in their portfolio. However, they did not allow for agents to move across regions. They find that, numerically, the short run dynamics of regional house price can be very different, depending on the region-specific adjustment cost of housing stock. On the other hand, Van Nieuwerburgh and Weill (2012) allow agents to be freely mobile. However, they assume agents to have linear preference non-durable consumption goods, and thus eliminating the need to insure for non-durable consumption risk. They focus on the house price dispersion and do not examine the house price dynamics.

model could be enriched by introducing household production.¹⁷ Among others, Davis and Martin (2009) and Dong et al. (2013) show, numerically, that introducing household production can improve the data-matching performance of a conventional business cycle model with housing. However, they do not explore the evolution of the house price dynamics analytically, and do not relate the house price dynamics to the dynamics of house-price-to-income ratio. Recently, Mian and Sufi (2010) and Mian et al. (2010), among others, provide micro-evidence that the mortgage supply and the political economy may be important in explaining why some counties have been affected more severely than others by the Great Recession. There are tractable models on how mortgage supply and house prices are both endogenously determined in a dynamic equilibrium setting, such as Chen et al. (2012), Chen and Leung (2008) and Jin et al. (2012). Chen et al. (2012) and Jin et al. (2012) both numerically matches some aspects of the housing market reasonably well. However, adding the political economy in any of these models may be very involved.¹⁸ We can thus only recognize these as limitations and leave them for future research.

¹⁷Greenwood et al (1995) show that the reduced form of models with household production that is subject to stochastic productivity shock may not be distinguishable from models with taste shock.

¹⁸Clearly, it is beyond the scope of this paper to explain the potential difficulties to extend an economic model to a political economy model. Among others, see Acemoglu and Robinson (2009), Banks and Hanushek (1995), Persson and Tabellini, 2002.

References

- [1] Acemoglu, D. and J. Robinson, 2009, *Economic Origins of Dictatorship and Democracy*, Cambridge: Cambridge University Press.
- [2] Banks, J. and E. Hanushek ed., 1995, *Modern Political Economy: Old Topics, New Directions*, Cambridge: Cambridge University Press.
- [3] Bartle, R. and D. Sherbert, 2011, *Introduction to Real Analysis*, 4th ed., New York: Wiley.
- [4] Capozza, D.; P. H. Hendershott, and C. Mack, 2004, An Anatomy of Price Dynamics in Illiquid Markets: Analysis and Evidence from Local Housing Markets, *Real Estate Economics*, 32, 1-32.
- [5] Chang, K. L.; N. K. Chen and C. K. Y. Leung, 2011, Monetary Policy, Term Structure and Real Estate Return: Comparing REIT, housing and stock, *Journal of Real Estate Finance and Economics*, 43, 221-257.
- [6] Chen, N. K.; Cheng, H. L. and C. S. Mao, 2012, House price, mortgage premium, and business fluctuations, *Economic Modelling*, 29(4), 1388-1398.
- [7] Chen, N. K. and C. K. Y. Leung, 2008, Asset Price Spillover, Collateral and Crises: with an Application to Property Market Policy, *Journal of Real Estate Finance and Economics*, 37, 351-385.
- [8] Cooley, T., ed., 1995, *Frontiers of Business Cycle Research*, Princeton: Princeton University Press.
- [9] Davidoff, T. 2013, Supply Elasticity and the Housing Cycle of the 2000s, forthcoming in *Real Estate Economics*.
- [10] Davis, M. and J. Heathcote, 2005, Housing and the business cycle, *International Economic Review*, 46, 751-784.
- [11] Davis, M. and F. Martin, 2009, Housing, home production, and the equity- and value-premium puzzles, *Journal of Housing Economics*, 18, 81-91.
- [12] Dong, J., F. Y. Kwan and C. K. Y. Leung, 2013, Comparing consumption-based asset pricing models, working paper.
- [13] Economist, 2011, Central Banks: a more complicated game, 17th February, <http://www.economist.com/node/18178251>.
- [14] Elaydi, S., 2005, *An Introduction to Difference Equations*, 3rd ed., New York: Springer.

- [15] Engle, R. and C. Granger, 1987, Co-integration and error correction: representation, estimation and testing, *Econometrica* 55, 251–76.
- [16] Farmer, R. E.A.; D. Waggoner and T. Zha, 2009, Understanding Markov-switching rational expectations models, *Journal of Economic Theory*, 144(5), 849-1867.
- [17] Fernandez-Villaverde, J., 2010, The econometrics of DSGE models, *SERIEs*, 1, 3-49.
- [18] Gallin, J., 2006, The long-run relationship between house prices and income: evidence from local housing markets, *Real Estate Economics*, 34, 417-438.
- [19] Glindro, E.; T. Subhanij, J. Szeto, and H. Zhu, 2011, Determinants of House Prices in Nine Asia-Pacific Economies, *International Journal of Central Banking*, 7, 163-204.
- [20] Granger, C., 1981, Some Properties of Time Series Data and Their Use in Econometric Model Specification, *Journal of Econometrics*, 121-130.
- [21] Greenwood, J. and Z. Hercowitz, 1991, The allocation of capital and time over the business cycle, *Journal of Political Economy*, 99, 1188-1214.
- [22] Greenwood, J.; R. Rogerson, and R. Wright, 1995, Household Production in Real Business Cycle Theory, in *Frontiers of Business Cycle Research*, ed. by T. F. Cooley, Princeton: Princeton University Press, 157-174.
- [23] Hamilton, J., 1994, *Time Series Analysis*, Princeton: Princeton University Press.
- [24] Hardy, G. H., J. E. Littlewood and G. Polya, 1934, *Inequalities*, Cambridge: Cambridge University Press.
- [25] Hercowitz, Z. and M. Sampson, 1991, Output, growth, the real wage, and employment fluctuations, *American Economic Review*, 81, 1215-37.
- [26] Iacoviello, M., 2005, House Prices, Borrowing Constraints, and Monetary Policy in the Business Cycle, *American Economic Review*, 95, 739-764.
- [27] Iacoviello, M., and S. Neri, 2010, Housing Market Spillovers: Evidence from an Estimated DSGE Model, *American Economic Journal: Macroeconomics*, 2, 125-164.
- [28] Iacoviello, M., and M. Pavan, 2012, Housing and Debt Over the Life Cycle and Over the Business Cycle, forthcoming in *Journal of Monetary Economics*.
- [29] Jin, Y.; C. K. Y. Leung and Z. Zeng, 2012, Real Estate, the External Finance Premium and Business Investment: A Quantitative Dynamic General Equilibrium Analysis, *Real Estate Economics*, 40(1), 167-195.
- [30] Jin, Y. and Z. Zeng, 2007, Real estate and optimal public policy in a credit-constrained economy, *Journal of Housing Economics*, 16(2), 143-166.

- [31] Kan, K.; Kwong, S. K. S.; C. K. Y. Leung, 2004, The dynamics and volatility of commercial and residential property prices: theory and evidence, *Journal of Regional Science*, 44(1), 95-123.
- [32] Keep, M., 2012, Regional house prices: affordability and income ratios, Standard Note SN/SG/1922, Library, House of Commons, Parliament of the United Kingdom.
- [33] King, R.; C. Plosser and S. Rebelo, 2002, Production, Growth and Business Cycles: Technical Appendix, *Computational Economics*, 20, 87-116.
- [34] King, R.; C. Plosser, J. Stock and M. Watson, 1991, Stochastic trends and economic fluctuations, *American Economic Review*, 81(4), 819-840.
- [35] Klyuev, V., 2008, What goes up must come down? house price dynamics in the United States, IMF Working Paper.
- [36] Kydland, F. E. and E. C. Prescott, 1996, The Computational Experiment: An Econometric Tool, *Journal of Economic Perspectives*, 10(1), 69-85.
- [37] Lau, S. H., 2002, Further inspection of the stochastic growth model by an analytical approach, *Macroeconomic Dynamics*, 6(5), 748-757.
- [38] Leung, C., 2004, Macroeconomics and Housing: a review of the literature, *Journal of Housing Economics*, 13, 249-267.
- [39] Leung, C. K. Y. and W. L. Teo, 2011, Should the optimal portfolio be region-specific? A multi-region model with monetary policy and asset price co-movements, *Regional Science and Urban Economics*, 41, 293-304.
- [40] Lucas, R., 1978, Asset Prices in an exchange economy, *Econometrica*, 46, 1426-45.
- [41] Malpezzi, S., 1999, A Simple Error Correction Model of House Prices, *Journal of Housing Economics*, 8, 27-62.
- [42] Mian, A. and A. Sufi, 2010, The Great Recession: lessons form microeconomic data, *American Economic Review: Papers and Proceedings*, 100, 51-56.
- [43] Mian, A.; A. Sufi and F. Trebbi, 2010, The political economy of the US mortgage default crisis, *American Economic Review*, 100, 1967-1998.
- [44] Ortalo-Magne, F. and S. Rady, 2006, Housing Market Dynamics: On the Contribution of Income Shocks and Credit Constraints, *Review of Economic Studies*, 73, 459-485.
- [45] Persson, T. and G. Tabellini, 2002, *Political Economics: Explaining Economic Policy (Zeuthen Lectures)*, Cambridge: MIT Press.

- [46] Ried, S. and H. Uhlig, 2009, The macroeconomics of real estate, Humboldt-Universität zu Berlin, working paper.
- [47] Sargent, T., 1979, *Macroeconomic Theory*, New York: Academic Press.
- [48] Sargent, T., 1987, *Dynamic Macroeconomic Theory*, Cambridge: Harvard University Press.
- [49] Slutsky, E., 1937, The Summation of Random Causes as the Source of Cyclic Processes, *Econometrica*, 5(2), 105-146.
- [50] Sowell, T., 2009, *The Housing Boom and Bust*, rev. ed., New York: Basic Books.
- [51] Steele, M., 2004, *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*. Cambridge: Cambridge University Press.
- [52] Stokey, N.; R. Lucas and E. Prescott, 1989, *Recursive Methods in Economic Dynamics*, Cambridge: Harvard University Press.
- [53] Taylor, J., 2009, *Getting Off Track: How Government Actions and Interventions Caused, Prolonged, and Worsened the Financial Crisis*, Calif.: Hoover Institution Press.
- [54] Van Nieuwerburgh, S., 2012, On Housing and the Macroeconomy (Research Agenda), *Economic Dynamics Newsletter*, 13. <http://www.economicdynamics.org/News261.htm#agenda>.
- [55] Van Nieuwerburgh, S. and P. O. Weill, 2012, Why has house price dispersion gone up? *Review of Economic Studies*, 77, 1567-1606.
- [56] Wall Street Journal, 2011, Linkage in Income, Home Prices Shifts (17th August).

A Proof

A.1 Proof of (11)

To solve the model, we need to first obtain the first order conditions. Let λ_{1t} , λ_{2t} and λ_{3t} denote the Lagrangian Multipliers of the constraints (9), (7) and (8), respectively. The first order conditions can be easily derived,

$$\lambda_{1t} = 1/C_t, \quad (26)$$

$$\lambda_{1t}P_t = \lambda_{3t}, \quad (27)$$

$$\lambda_{1t} = \lambda_{2t}\delta \left(\frac{K_{t+1}}{I_t^K} \right), \quad (28)$$

$$\lambda_{1t} = \lambda_{3t}\gamma \left(\frac{H_{t+1} - H_t^S}{I_t^H} \right), \quad (29)$$

$$w_L(N_t)^\eta = \lambda_{1t}(1 - \alpha) \left(\frac{Y_t}{N_t} \right), \quad (30)$$

$$\lambda_{3t} = \beta E_t \left[\frac{w_H}{H_{t+1}} + \lambda_{3,t+1}(1 - \gamma) \left(\frac{H_{t+2} - H_{t+1}^S}{H_{t+1}} \right) \right], \quad (31)$$

$$\lambda_{2t} = \beta E_t \left[\lambda_{2,t+1}(1 - \delta) \frac{K_{t+2}}{K_{t+1}} + \lambda_{1,t+1}\alpha \left(\frac{Y_{t+1}}{K_{t+1}} \right) \right]. \quad (32)$$

First, notice that the model economy satisfies the standard conditions in Stokey, Lucas and Prescott (1989, chapter 9), it is easy to see that the equilibrium is unique, and hence justifies the approach of Sargent (1987), which is to conjecture that (11) is indeed correct and verify that ϕ_c , ϕ_K , ϕ_H and η_N are indeed constant.

We first combine (31) with (10), and solve recursively, we have

$$\lambda_{3t}H_{t+1} = \frac{\beta w_H}{1 - \beta(1 - \gamma)}, \quad (33)$$

if we have the following transversality condition satisfied,

$$\lim_{t \rightarrow \infty} \beta^t \lambda_{3t}H_{t+1} = 0. \quad (34)$$

Notice also that $0 < \beta, \gamma < 1$, Thus $\lambda_{3t}H_{t+1} > 0$. We then combine (26), (29) with (33), we get

$$\phi_H = \frac{\beta w_H \gamma}{1 - \beta(1 - \gamma)} \phi_c. \quad (35)$$

By the same token, if we impose another transversality condition,

$$\lim_{t \rightarrow \infty} \beta^t \lambda_{2t}K_{t+1} = 0, \quad (36)$$

and combine (26), (28), (32), we will have

$$\phi_K = \frac{\beta\alpha\gamma}{1 - \beta(1 - \delta)}. \quad (37)$$

Notice that $\phi_K > 0$ as $\beta\alpha\gamma > 0$, and $\beta(1 - \delta) < 1$. And since ϕ_K represents a share of the output, we need $\phi_K < 1$. It means that the third condition we need to impose is

$$\begin{aligned} \beta\alpha\gamma &< 1 - \beta(1 - \delta) \\ \text{or, } \beta\alpha\gamma + \beta(1 - \delta) &< 1. \end{aligned} \quad (38)$$

By (30), we have

$$\begin{aligned} (N_t)^{\eta+1} &= \frac{(1 - \alpha)}{w_L} (\lambda_{1t} Y_t) \\ &= \frac{(1 - \alpha)}{w_L} \frac{1}{\phi_c}. \end{aligned}$$

Thus, if ϕ_c is a constant, N_t is also a constant.

Now, by (9) and (10), we have

$$\phi_c + \phi_K + \phi_H = 1.$$

Combine it with (35) and (37), we get

$$\phi_c = \frac{1 - \beta(1 - \delta) - \beta\alpha\gamma}{1 - \beta(1 - \delta) + \beta w_H \gamma},$$

which is clearly a constant. Clearly, $\phi_c < 1$. By (38), we also have $\phi_c > 0$. By (35), we have

$$\phi_H = \frac{\beta w_H \gamma}{1 - \beta(1 - \gamma)} \frac{1 - \beta(1 - \delta) - \beta\alpha\gamma}{1 - \beta(1 - \delta) + \beta w_H \gamma}.$$

Again, by (38), it is clear that $\phi_H > 0$. And if we write $\beta w_H \gamma \equiv a_\phi$, $[1 - \beta(1 - \gamma)] \equiv b_\phi$, $\beta\alpha\gamma \equiv c_\phi$, $[1 - \beta(1 - \delta)] \equiv d_\phi$, and $a_\phi, b_\phi, d_\phi > 0$, then

$$\phi_H = \frac{a_\phi d_\phi - c_\phi}{b_\phi d_\phi + a_\phi}.$$

Thus, for $\phi_H < 1$, we need

$$(\delta - \gamma) < \frac{a_\phi c_\phi + b_\phi d_\phi}{\beta}. \quad (39)$$

Thus, for the conjecture to be valid, we need to impose (34), (36) and (38), (39).

A.2 Proof of (13), (14)

We will first provide the proof of (13), which is relatively easy. Taking natural log of (5), (7), and (11), we have

$$\begin{aligned} y_t &= a_t + \alpha k_t + b_y, \\ k_{t+1} &= b_k + (1 - \delta) k_t + \delta i_t^k, \\ i_t^k &= b_{ik} + y_t. \end{aligned}$$

The first can be re-written as

$$y_{t+1} - \alpha k_{t+1} = b_y + a_{t+1}.$$

The last two can be combined as

$$\begin{aligned} k_{t+1} &= b_k + (1 - \delta) k_t + \delta (b_{ik} + y_t) \\ &= b'_k + (1 - \delta) k_t + \delta y_t, \end{aligned}$$

where $b'_k \equiv b_k + \delta b_{ik}$. Thus, we have

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} b_y \\ b'_k \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta & (1 - \delta) \end{pmatrix} \begin{pmatrix} y_t \\ k_t \end{pmatrix} + \begin{pmatrix} a_{t+1} \\ 0 \end{pmatrix}, \quad (40)$$

which is in a form similar to structural VAR.

Now notice that $\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. Thus, (40) can be written as

$$\begin{pmatrix} y_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} b'_y \\ b'_k \end{pmatrix} + \begin{pmatrix} \alpha\delta & \alpha(1 - \delta) \\ \delta & (1 - \delta) \end{pmatrix} \begin{pmatrix} y_t \\ k_t \end{pmatrix} + \begin{pmatrix} a_{t+1} \\ 0 \end{pmatrix},$$

where $b'_y = b_y + \alpha b'_k$. This also completes the proof of (13).

To prove (14), we need some more notations. Let L be the Lag operator, $L^i x_t = x_{t-i}$, $i = 1, 2, \dots$, and $Lc = c$, for all constant c (see Sargent, 1987). Then (13) can be rewritten as

$$\vec{y}_t = M_0 + M_1(L) \vec{y}_t + \vec{u}_t,$$

where

$$M_1(L) = \begin{pmatrix} \alpha\delta L & \alpha(1 - \delta)L \\ \delta L & (1 - \delta)L \end{pmatrix}.$$

It follows that

$$\vec{y}_t = (I - M_1(L))^{-1} M_0 + (I - M_1(L))^{-1} \vec{u}_t,$$

where I is the identity matrix. By Cramer's Rule,

$$(I - M_1(L))^{-1} = [1 - (1 - \delta + \alpha\delta)L]^{-1} \begin{pmatrix} 1 - (1 - \delta)L & \alpha(1 - \delta)L \\ \delta L & 1 - \alpha\delta L \end{pmatrix},$$

and recall that $[1 - (1 - \delta + \alpha\delta)L]^{-1}$ can be understood as a shorthand for a polynomial,

$$[1 - (1 - \delta + \alpha\delta)L]^{-1} = \sum_{i=0}^{\infty} [(1 - \delta + \alpha\delta)L]^i.$$

And since $M_0 = \begin{pmatrix} b'_y \\ b'_k \end{pmatrix}$,

$$\begin{aligned} & (I - M_1(L))^{-1} M_0 \\ &= (\delta(1 - \alpha))^{-1} \begin{pmatrix} \delta b'_y + \alpha(1 - \delta)b'_k \\ \delta b'_y + (1 - \alpha\delta)b'_k \end{pmatrix} \equiv \begin{pmatrix} b''_y \\ b''_k \end{pmatrix}. \end{aligned}$$

Similarly, as $\vec{u}_t = \begin{pmatrix} a_t \\ 0 \end{pmatrix}$, we have

$$\begin{aligned} y_t &= b''_y + a_t + (\alpha\delta) \sum_{i=0}^{\infty} (1 - \delta + \alpha\delta)^i a_{t-1-i}, \\ k_t &= b''_k + \delta \sum_{i=0}^{\infty} (1 - \delta + \alpha\delta)^i a_{t-1-i}. \end{aligned} \quad (41)$$

Notice that the impact from a_t to y_t is “one-to-one,” whereas that from a_t to k_t is zero. It is because the business capital is a state variable and a_t can only influence k_{t+j} , $j = 1, 2, \dots$

To obtain an expression of h_t in terms of $\{a_t\}$, recall from (12) that $h_{t+1} = b_h + (1 - \gamma)h_t + \gamma i_t^h$, and $i_t^h = b_{ih} + y_t$. It means that

$$h_{t+1} = b'_h + (1 - \gamma)h_t + \gamma y_t, \quad (42)$$

where $b'_h \equiv b_h + \gamma b_{ih}$. Solving (42) recursively, we get

$$\begin{aligned} h_t &= b'_h/\gamma + [1 - (1 - \gamma)L]^{-1} \gamma L y_t \\ &= b_h^{(2)} + \gamma \sum_{i=0}^{\infty} (1 - \gamma)^i y_{t-1-i}, \end{aligned} \quad (43)$$

where $b_h^{(2)}$ is a constant. Combining (43) with (41) will deliver

$$\begin{aligned} h_t &= b_h^{(2)} + \gamma \sum_{i=0}^{\infty} (1 - \gamma)^i \left(b''_y + a_{t-1-i} + (\alpha\delta) \sum_{j=0}^{\infty} (1 - \delta + \alpha\delta)^j a_{t-2-i-j} \right) \\ &= b''_h + \gamma a_{t-1} + \gamma \sum_{i=0}^{\infty} \delta_h(i) a_{t-2-i}, \end{aligned}$$

where b_h'' is a constant,

$$\delta_h(i) = (1 - \gamma)^{i+1} + (\alpha\delta) \sum_{j=0}^i (1 - \gamma)^{i-j} (1 - \delta + \alpha\delta)^j, \quad (44)$$

with $\sum_{j=0}^0 (1 - \gamma)^{i-j} (1 - \delta + \alpha\delta)^j \equiv 1$. And it is the last expression of (14).

In fact, based on (14), we can relate the output growth to the productivity growth more explicitly.

Lemma 1 *The growth rate of the output can be expressed in the following way:*

$$dy_{t+1} = da_{t+1} + (\alpha\delta) a_t - (\alpha\delta^2) (1 - \alpha) \sum_{i=0}^{\infty} (1 - \delta + \alpha\delta)^i a_{t-1-i}, \quad (45)$$

where dy_{t+1} is the output growth factor,

$$dy_{t+1} \equiv (y_{t+1} - y_t) = \ln(Y_{t+1}/Y_t),$$

and da_{t+1} is the productivity growth factor,

$$da_{t+1} \equiv (a_{t+1} - a_t) = \ln(A_{t+1}/A_t).$$

The proof of the proposition, (45), follows immediately from (14). First, we update the formula for y_{t+1} . We then take the different between y_{t+1} and y_t , and the result will be delivered.

A.3 Proof of (15)

To address the stationarity of the output, we first recall a well known result from Wold.

Theorem 2 *Wold's Decomposition Theorem (Hamilton, 1994, p.109). Any covariance-stationary process Ψ_t can be represented in the form*

$$\Psi_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \kappa_t,$$

where the finite square summation condition holds,

$$\sum_{j=0}^{\infty} |\psi_j|^2 < \infty, \quad (46)$$

and the term ε_t is a white noise and represents the forecasting error of Ψ_t ,

$$\varepsilon_t \equiv \Psi_t - E(\Psi_t | \Psi_{t-1}, \Psi_{t-2}, \dots),$$

with κ_t un-correlated with ε_{t-j} for any j .

Now, recall from equation (6) that

$$a_t = \rho a_{t-1} + \varepsilon_t,$$

where $a_t = \ln(A_t)$, $0 < \rho \leq 1$, and ε_t is a white noise process. Clearly, for $0 < \rho < 1$,

$$a_t = \varepsilon_t + \sum_{j=1}^{\infty} \rho^j \varepsilon_{t-j}. \quad (47)$$

and hence (14) can be written as

$$\begin{aligned} y_t &= b_y'' + \varepsilon_t + \sum_{i=1}^{\infty} \left(\rho^i + \left((\alpha\delta) \frac{\rho^i - (1 - \delta + \alpha\delta)^i}{\rho - (1 - \delta + \alpha\delta)} \right) \right) \varepsilon_{t-i} \\ &= b_y'' + \varepsilon_t + \sum_{i=1}^{\infty} \left(\rho^i \frac{\rho - 1 + \delta}{\rho - (1 - \delta + \alpha\delta)} + \frac{(\alpha\delta)}{\rho - (1 - \delta + \alpha\delta)} (1 - \delta + \alpha\delta)^i \right) \varepsilon_{t-i}. \end{aligned} \quad (48)$$

Based on (48) and the Wold's theorem, the stationarity of y_t will depend on whether the following expression is finite or not,

$$\sum_{i=1}^{\infty} \left| \rho^i \frac{\rho - 1 + \delta}{\rho - (1 - \delta + \alpha\delta)} + \frac{(\alpha\delta)}{\rho - (1 - \delta + \alpha\delta)} (1 - \delta + \alpha\delta)^i \right|^2. \quad (49)$$

A few observations are in order. First, since $0 < \rho$, $(1 - \delta + \alpha\delta) < 1$, $0 < \rho^i$, $(1 - \delta + \alpha\delta)^i < 1$, $\forall i \geq 1$. Second, define $\rho_\alpha \equiv \max \left\{ \left| \frac{\rho - 1 + \delta}{\rho - (1 - \delta + \alpha\delta)} \right|, \left| \frac{(\alpha\delta)}{\rho - (1 - \delta + \alpha\delta)} \right| \right\}$. Clearly, ρ_α is real and finite, and by construction, $\rho_\alpha > 0$. Similarly, define $\rho_\delta \equiv \max \{ \rho, (1 - \delta + \alpha\delta) \}$. Clearly, since $0 < \rho$, $(1 - \delta + \alpha\delta) < 1$, $0 < \rho_\delta < 1$. Thus, (49) can be re-written as,

$$\begin{aligned} & \sum_{i=1}^{\infty} \left| \rho^i \frac{\rho - 1 + \delta}{\rho - (1 - \delta + \alpha\delta)} + \frac{(\alpha\delta)}{\rho - (1 - \delta + \alpha\delta)} (1 - \delta + \alpha\delta)^i \right|^2 \\ & < \sum_{i=1}^{\infty} (\rho_\delta)^{2i} \left| \frac{\rho - 1 + \delta}{\rho - (1 - \delta + \alpha\delta)} + \frac{(\alpha\delta)}{\rho - (1 - \delta + \alpha\delta)} \right|^2 \\ & < (2\rho_\alpha)^2 \sum_{i=1}^{\infty} (\rho_\delta)^{2i}. \end{aligned}$$

Notice that $(2\rho_\alpha)^2$ is a positive constant. By Bartle and Sherbert (2011), it is easy to show that the series $\left\{ (\rho_\delta)^{2i} \right\}$ is Cauchy. In fact, the infinite sum $\sum_{i=1}^{\infty} (\rho_\delta)^{2i} = (\rho_\delta)^2 / (1 - (\rho_\delta)^2)$, which is finite. Hence, y_t is stationary. And it proves (15).

A.4 Proof of (16)

The proof of this proposition is not that difficult. Combining (26), (27), (31), (10), we have

$$\lambda_{1t}P_t = \beta E_t \left[\frac{w_H}{H_{t+1}} + \lambda_{1,t+1}P_{t+1} (1 - \gamma) \left(\frac{H_{t+2}}{H_{t+1}} \right) \right]. \quad (50)$$

We conjecture that

$$\lambda_{1t}P_t H_{t+1} = x, \forall t, \quad (51)$$

where x is some constant. Under this conjecture, (50) can be reduced to

$$x = \frac{\beta w_H}{1 - \beta(1 - \gamma)}, \quad (52)$$

and the right hand side is indeed a constant. Now combining (51), (52) with (26), (11), we get

$$H_{t+1} = \frac{\beta w_H \phi_c}{1 - \beta(1 - \gamma)} \frac{Y_t}{P_t},$$

or, in log form,

$$h_{t+1} = b_p + y_t - p_t,$$

where b_p is a constant, and is equivalent to

$$p_t = b_p + y_t - h_{t+1},$$

which is the first statement of (16).

Now recall (43) that

$$\begin{aligned} h_t &= b'_h / \gamma + [1 - (1 - \gamma)L]^{-1} \gamma L y_t \\ &= b_h^{(2)} + \gamma \sum_{i=0}^{\infty} (1 - \gamma)^i y_{t-1-i}, \end{aligned}$$

where $b_h^{(2)}$ is a constant term. Thus, the term $(y_t - h_{t+1})$ can be written as

$$\begin{aligned} y_t - h_{t+1} &= y_t - \left[b_h^{(2)} + \gamma \sum_{i=0}^{\infty} (1 - \gamma)^i y_{t-i} \right] \\ &= -b_h^{(2)} + (1 - \gamma)y_t - \gamma \sum_{i=1}^{\infty} (1 - \gamma)^i y_{t-i} \\ &= -b_h^{(2)} + (1 - \gamma) \left[y_t - \gamma \sum_{i=1}^{\infty} (1 - \gamma)^{i-1} y_{t-i} \right]. \end{aligned}$$

This implies that

$$p_t = b'_p + (1 - \gamma) \left[y_t - \gamma \sum_{i=1}^{\infty} (1 - \gamma)^{i-1} y_{t-i} \right]. \quad (53)$$

Notice that

$$\gamma \sum_{i=1}^{\infty} (1-\gamma)^{i-1} = 1.$$

Hence, we can write

$$y_t = \gamma \sum_{i=1}^{\infty} (1-\gamma)^{i-1} y_t,$$

and (53) becomes

$$\begin{aligned} p_t &= b'_p + (1-\gamma) \left[\gamma \sum_{i=1}^{\infty} (1-\gamma)^{i-1} (y_t - y_{t-i}) \right] \\ &= b'_p + \gamma \sum_{i=1}^{\infty} (1-\gamma)^i (y_t - y_{t-i}), \end{aligned}$$

which is the second statement of (16).

A.5 Proof of (17)

The proof proceeds in a few steps. First, we need to obtain an expression of the house price in terms of the productivity shocks $\{a_t\}$. Notice that by (16), we have

$$p_t = b'_p + (1-\gamma) \left[y_t - \gamma \sum_{i=1}^{\infty} (1-\gamma)^{i-1} y_{t-i} \right],$$

and by (14), we have

$$y_t = b''_y + a_t + (\alpha\delta) \sum_{i=0}^{\infty} (1-\delta + \alpha\delta)^i a_{t-1-i}.$$

Together, we have

$$\begin{aligned} p_t &= b''_p + (1-\gamma) a_t + (1-\gamma) [\alpha\delta - \gamma] a_{t-1} \\ &\quad + (1-\gamma) [(\alpha\delta)(1-\delta + \alpha\delta) - \gamma(\alpha\delta) - \gamma(1-\gamma)] a_{t-2} \\ &\quad + (1-\gamma) \left\{ (\alpha\delta)(1-\delta + \alpha\delta)^2 - \gamma(\alpha\delta)[(1-\delta + \alpha\delta) + (1-\gamma)] \right. \\ &\quad \left. - \gamma(1-\gamma)^2 \right\} a_{t-3} \\ &\quad + \dots \\ &\quad + (1-\gamma) \left\{ (\alpha\delta)(1-\delta + \alpha\delta)^{i-1} - \gamma(\alpha\delta) \sum_{j=0}^{i-2} [(1-\delta + \alpha\delta)^{i-2-j} (1-\gamma)^j] \right. \\ &\quad \left. - \gamma(1-\gamma)^{i-1} \right\} a_{t-i} \\ &\quad + \dots \end{aligned}$$

where b_p'' is some constant. Observe that for any a, b, n , $\sum_{i=0}^n a^{n-i}b^i = [a^{n+1} - b^{n+1}] / (a - b)$. Thus,

$$\begin{aligned}
p_t &= b_p'' + (1 - \gamma) a_t + (1 - \gamma) [\alpha\delta - \gamma] a_{t-1} \\
&\quad + (1 - \gamma) [(\alpha\delta)(1 - \delta + \alpha\delta) - \gamma(\alpha\delta) - \gamma(1 - \gamma)] a_{t-2} \\
&\quad + \sum_{i=3}^{\infty} (1 - \gamma) \left\{ (\alpha\delta)(1 - \delta + \alpha\delta)^{i-1} - \gamma(\alpha\delta) \frac{(1 - \delta + \alpha\delta)^{i-1} - (1 - \gamma)^{i-1}}{(1 - \delta + \alpha\delta) - (1 - \gamma)} \right. \\
&\quad \left. - \gamma(1 - \gamma)^{i-1} \right\} a_{t-i} \\
&= b_p'' + (1 - \gamma) a_t + (1 - \gamma) [\alpha\delta - \gamma] a_{t-1} \\
&\quad + (1 - \gamma) [(\alpha\delta)(1 - \delta + \alpha\delta) - \gamma(\alpha\delta) - \gamma(1 - \gamma)] a_{t-2} \\
&\quad + \sum_{i=3}^{\infty} (1 - \gamma) \left\{ (\alpha\delta)(1 - \delta + \alpha\delta)^{i-1} \left(1 - \frac{\gamma}{(1 - \delta + \alpha\delta) - (1 - \gamma)} \right) \right. \\
&\quad \left. - \gamma(1 - \gamma)^{i-1} \left(1 - \frac{(\alpha\delta)}{(1 - \delta + \alpha\delta) - (1 - \gamma)} \right) \right\} a_{t-i}
\end{aligned}$$

Notice that

$$\begin{aligned}
&\left\{ (\alpha\delta)(1 - \delta + \alpha\delta)^{i-1} - \gamma(\alpha\delta) \frac{(1 - \delta + \alpha\delta)^{i-1} - (1 - \gamma)^{i-1}}{(1 - \delta + \alpha\delta) - (1 - \gamma)} - \gamma(1 - \gamma)^{i-1} \right\} \\
&= (\alpha\delta) \left(\frac{\delta - \alpha\delta}{\delta - \alpha\delta - \gamma} \right) (1 - \delta + \alpha\delta)^{i-1} - \gamma \left(\frac{\delta - \gamma}{\delta - \alpha\delta - \gamma} \right) (1 - \gamma)^{i-1}.
\end{aligned}$$

And for future reference, we define a new term $\delta^p(i)$,

$$\delta^p(i) \equiv (1 - \gamma) \left[(\alpha\delta) \left(\frac{\delta - \alpha\delta}{\delta - \alpha\delta - \gamma} \right) (1 - \delta + \alpha\delta)^{i-1} - \gamma \left(\frac{\delta - \gamma}{\delta - \alpha\delta - \gamma} \right) (1 - \gamma)^{i-1} \right]. \quad (54)$$

We further observe that if we use the following notations,

$$\begin{aligned}
x &= (1 - \gamma) \\
z &= (1 - \delta + \alpha\delta) \\
\mathcal{A} &= (1 - \gamma) (\alpha\delta) \left(\frac{\delta - \alpha\delta}{\delta - \alpha\delta - \gamma} \right) \\
\mathcal{B} &= (1 - \gamma) \gamma \left(\frac{\delta - \gamma}{\delta - \alpha\delta - \gamma} \right)
\end{aligned} \quad (55)$$

Notice that

$$0 < x, z < 1. \quad (56)$$

Then,

$$\delta^p(i) = \mathcal{A}(z)^{i-1} - \mathcal{B}(x)^{i-1}. \quad (57)$$

Hence

$$\begin{aligned}
p_t &= b_p'' + (1 - \gamma) a_t + (1 - \gamma) [\alpha\delta - \gamma] a_{t-1} \\
&\quad + (1 - \gamma) [(\alpha\delta)(1 - \delta + \alpha\delta) - \gamma(\alpha\delta) - \gamma(1 - \gamma)] a_{t-2} \\
&\quad + \sum_{i=3}^{\infty} \delta^p(i) a_{t-i}.
\end{aligned} \tag{58}$$

To verify the stationarity of p_t , it suffices to check whether $\sum_{i=3}^{\infty} \delta^p(i) a_{t-i}$ is stationary or not. Again, recall from equation (6) that

$$a_t = \rho a_{t-1} + \varepsilon_t,$$

where $a_t = \ln(A_t)$, $0 < \rho \leq 1$, and ε_t is a white noise process. Clearly, when $0 < \rho < 1$, we have $a_t = \varepsilon_t + \sum_{j=1}^{\infty} \rho^j \varepsilon_{t-j}$, and hence

$$\begin{aligned}
&\sum_{i=3}^{\infty} \delta^p(i) a_{t-i} \\
&= \sum_{i=3}^{\infty} \left(\sum_{j=3}^i \rho^{i-j} \delta^p(j) \right) \varepsilon_{t-i}.
\end{aligned}$$

By the Wold's decomposition theorem, p_t is stationary if $\sum_{i=3}^{\infty} \left| \sum_{j=3}^i \rho^{i-j} \delta^p(j) \right|^2 < \infty$. As we discussed before, $\sum_{i=3}^{\infty} \left| \sum_{j=3}^i \delta^p(j) \right|^2 < \infty$ if and only if $\forall \epsilon > 0, \exists N > 0$, such that $\forall n > m > N, \sum_{i=m}^n \left| \sum_{j=3}^i \delta^p(j) \right|^2 < \epsilon$ (see Bartle and Sherbert, 2011).

By (57), (55),

$$\begin{aligned}
&\sum_{j=3}^i \rho^{i-j} \delta^p(j) \\
&= \mathcal{A} \left(z^2 \frac{\rho^{i-2} - z^{i-2}}{\rho - z} \right) - \mathcal{B} \left(x^2 \frac{\rho^{i-2} - x^{i-2}}{\rho - x} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left| \sum_{j=3}^i \rho^{i-j} \delta^p(j) \right|^2 \\
&= \left| \mathcal{A} \left(z^2 \frac{\rho^{i-2} - z^{i-2}}{\rho - z} \right) - \mathcal{B} \left(x^2 \frac{\rho^{i-2} - x^{i-2}}{\rho - x} \right) \right|^2 \\
&\leq \left| \mathcal{A} \left(z^2 \frac{\rho^{i-2} - z^{i-2}}{\rho - z} \right) \right|^2 + \left| \mathcal{B} \left(x^2 \frac{\rho^{i-2} - x^{i-2}}{\rho - x} \right) \right|^2 \\
&\quad + 2 \left| \mathcal{A} \left(z^2 \frac{\rho^{i-2} - z^{i-2}}{\rho - z} \right) \right| \left| \mathcal{B} \left(x^2 \frac{\rho^{i-2} - x^{i-2}}{\rho - x} \right) \right|
\end{aligned}$$

Let $\xi^p = \max \left\{ \left| \mathcal{A} \left(\frac{z^2}{\rho-z} \right) \right|, \left| \mathcal{B} \left(\frac{x^2}{\rho-x} \right) \right| \right\}$. Clearly, ξ^p is a real number, does not depend on i and $\xi^p > 0$. Thus,

$$\begin{aligned} & \left| \sum_{j=3}^i \rho^{i-j} \delta^p(j) \right|^2 \\ & \leq \left| \mathcal{A} \left(z^2 \frac{\rho^{i-2} - z^{i-2}}{\rho-z} \right) \right|^2 + \left| \mathcal{B} \left(x^2 \frac{\rho^{i-2} - x^{i-2}}{\rho-x} \right) \right|^2 \\ & \quad + 2 \left| \mathcal{A} \left(z^2 \frac{\rho^{i-2} - z^{i-2}}{\rho-z} \right) \right| \left| \mathcal{B} \left(x^2 \frac{\rho^{i-2} - x^{i-2}}{\rho-x} \right) \right| \\ & < (\xi^p)^2 \left\{ |\rho^{i-2} - z^{i-2}|^2 + |\rho^{i-2} - x^{i-2}|^2 + 2 |\rho^{i-2} - z^{i-2}| |\rho^{i-2} - x^{i-2}| \right\} \end{aligned}$$

Notice that $0 < x, z, \rho < 1$. It follows that as $i \rightarrow \infty$, $x^{i-2}, \rho^{i-2}, z^{i-2} \rightarrow 0$. Let $\rho^p = \max \{x, z, \rho\}$. Thus, $\forall \epsilon > 0, \exists N > 0, \forall i > N, (\rho^p)^{i-2} < \sqrt{\epsilon}/(4\xi^p)$. Thus, $|\rho^{i-2} - z^{i-2}| < 2(\rho^p)^{i-2} < \sqrt{\epsilon}/(2\xi^p)$. It follows that $|\rho^{i-2} - z^{i-2}|^2 < \epsilon/(2\xi^p)^2$. By the same token, $|\rho^{i-2} - x^{i-2}|^2 < \epsilon/(2\xi^p)^2$, and $2 |\rho^{i-2} - z^{i-2}| |\rho^{i-2} - x^{i-2}| < \epsilon/(2(\xi^p)^2)$. Therefore,

$$\begin{aligned} & \left| \sum_{j=3}^i \rho^{i-j} \delta^p(j) \right|^2 \\ & < (\xi^p)^2 \left\{ |\rho^{i-2} - z^{i-2}|^2 + |\rho^{i-2} - x^{i-2}|^2 + 2 |\rho^{i-2} - z^{i-2}| |\rho^{i-2} - x^{i-2}| \right\} \\ & \leq (\xi^p)^2 \left\{ \frac{\epsilon}{4(\xi^p)^2} + \frac{\epsilon}{4(\xi^p)^2} + \frac{\epsilon}{2(\xi^p)^2} \right\} = \epsilon. \end{aligned}$$

Consequently, $\sum_{i=m}^n \left| \sum_{j=3}^i \rho^{i-j} \delta^p(j) \right|^2 < \sum_{i=m}^n \epsilon = (n-m) * \epsilon$. This implies that p_t is stationary. This completes the proof of (17).

A.6 Proof of (18)

From (12), we have

$$h_{t+1} = b'_h + (1 - \gamma) h_t + \gamma y_t,$$

where b'_h is a constant.

Recall from (16) that

$$p_t = b_p + y_t - h_{t+1}.$$

Thus, we have

$$\begin{aligned} p_t &= b_p + y_t - (b'_h + (1 - \gamma) h_t + \gamma y_t) \\ &= b_p + y_t - [b'_h + (1 - \gamma) (b_p + y_{t-1} - p_{t-1}) + \gamma y_t] \end{aligned}$$

which implies that

$$(y_t - p_t) = \bar{b}_p + (1 - \gamma)(y_{t-1} - p_{t-1}) + \gamma y_t, \quad (59)$$

where $\bar{b}_p = (b'_h - b_p)$ is a constant. If we subtract γp_t from both sides, we have

$$(y_t - p_t) - \gamma p_t = \bar{b}_p + (1 - \gamma)(y_{t-1} - p_{t-1}) + \gamma(y_t - p_t),$$

or

$$(1 - \gamma)(y_t - p_t) - \gamma p_t = \bar{b}_p + (1 - \gamma)(y_{t-1} - p_{t-1}). \quad (60)$$

We now update the last expression,

$$(1 - \gamma)(y_{t+1} - p_{t+1}) - \gamma p_{t+1} = \bar{b}_p + (1 - \gamma)(y_t - p_t). \quad (61)$$

Combining (60) and (61), we have

$$\begin{aligned} dp_{t+1} &\equiv (p_{t+1} - p_t) \\ &= \frac{(1 - \gamma)}{\gamma} \{[\varkappa_{t+1} - \varkappa_t] - [\varkappa_t - \varkappa_{t-1}]\}, \end{aligned}$$

where

$$\varkappa_t \equiv y_t - p_t,$$

which is the income-house price ratio.

A.7 Proof of (21)

Recall (18) that

$$dp_{t+1} = \frac{(1 - \gamma)}{\gamma} \{[\varkappa_{t+1} - \varkappa_t] - [\varkappa_t - \varkappa_{t-1}]\},$$

where $dp_{t+1} \equiv (p_{t+1} - p_t)$, $\varkappa_t \equiv y_t - p_t$. The expression above can be re-written as

$$p_{t+1} - (2 - \gamma)p_t + (1 - \gamma)p_{t-1} = (1 - \gamma)\{dy_{t+1} - dy_t\}, \quad (62)$$

where $dy_{t+1} \equiv (y_{t+1} - y_t)$, which is (21).

Following Elaydi (2005) or other standard textbooks, the characteristic equation for the left hand side is simply,

$$\begin{aligned} &x^2 - (2 - \gamma)x + (1 - \gamma) \\ &= (x - (1 - \gamma))(x - 1). \end{aligned}$$

Thus, the solution for the *homogeneous equation* associated with (62) is of the form

$$p_t = (\mathcal{A}_1 + \mathcal{A}_2 t)(1 - \gamma)^t,$$

where $\mathcal{A}_1, \mathcal{A}_2$ are parameters. If the right hand side of (62) were a polynomial of t , then we can solve the general solution by using textbook methods. Unfortunately, the right hand side of (62) is $(1 - \gamma) \{dy_{t+1} - dy_t\}$, which is *stochastic* and rather complicated. Hence, a general, closed form solution for (62) may not be available.

To see that, let us separate the two cases, which are $\rho = 1$ and $0 < \rho < 1$.

- Case 1: $\rho = 1$.

In this case, by (74), we have

$$\begin{aligned} dy_{t+1} &= y_{t+1} - y_t \\ &= \varepsilon_{t+1} + (\alpha\delta) \sum_{j=1}^{\infty} (1 - \delta + \alpha\delta)^{j-1} \varepsilon_{t+1-j}. \end{aligned}$$

Similar, $dy_t = \varepsilon_t + (\alpha\delta) \sum_{j=1}^{\infty} (1 - \delta + \alpha\delta)^{j-1} \varepsilon_{t-j}$. Thus,

$$\begin{aligned} dy_{t+1} - dy_t &= \varepsilon_{t+1} - (1 - \alpha\delta) \varepsilon_t - (\alpha\delta^2) (1 - \alpha) \sum_{j=1}^{\infty} (1 - \delta + \alpha\delta)^{j-1} \varepsilon_{t-j}. \end{aligned}$$

- Case 2: $0 < \rho < 1$.

In this case, by (48), we have

$$\begin{aligned} dy_{t+1} &= y_{t+1} - y_t \\ &= \varepsilon_{t+1} + (\mathcal{D}_1\rho + \mathcal{D}_2z - 1) \varepsilon_t - \sum_{j=1}^{\infty} (\mathcal{D}_1\rho^j (1 - \rho) + \mathcal{D}_2z^j (1 - z)) \varepsilon_{t-j}, \end{aligned}$$

where $\mathcal{D}_1 \equiv \frac{\rho-1+\delta}{\rho-(1-\delta+\alpha\delta)}$, $\mathcal{D}_2 \equiv \frac{\alpha\delta}{\rho-(1-\delta+\alpha\delta)}$, $z = (1 - \delta + \alpha\delta)$. Clearly, $0 < \rho, z < 1$.

Similar, $dy_t = \varepsilon_t + (\mathcal{D}_1\rho + \mathcal{D}_2z - 1) \varepsilon_{t-1} - \sum_{j=1}^{\infty} (\mathcal{D}_1\rho^j (1 - \rho) + \mathcal{D}_2z^j (1 - z)) \varepsilon_{t-1-j}$.

Thus,

$$\begin{aligned} dy_{t+1} - dy_t &= \varepsilon_{t+1} - (\mathcal{D}_1\rho + \mathcal{D}_2z - 2) \varepsilon_t \\ &\quad + (1 - 2\mathcal{D}_1\rho - 2\mathcal{D}_2z + \mathcal{D}_1\rho^2 + \mathcal{D}_2z^2) \varepsilon_{t-1} \\ &\quad + \sum_{j=2}^{\infty} (\mathcal{D}_1\rho^{j-1} (1 - \rho)^2 + \mathcal{D}_2z^{j-2} (1 - z)^2) \varepsilon_{t-j}. \end{aligned}$$

A.8 Proof of (23)

Recall from equation (16), we have

$$y_t - p_t = -b_p + h_{t+1},$$

and since from (14), we know that

$$\begin{aligned} h_t &= b_h^{(2)} + \gamma \sum_{i=0}^{\infty} (1-\gamma)^i \left(b_y'' + a_{t-1-i} + (\alpha\delta) \sum_{j=0}^{\infty} (1-\delta + \alpha\delta)^j a_{t-2-i-j} \right) \\ &= b_h'' + \gamma a_{t-1} + \gamma \sum_{i=0}^{\infty} \delta_h(i) a_{t-2-i}, \end{aligned}$$

where b_h'' is a constant,

$$b_h'' = b_h^{(2)} + b_y'',$$

and by (44),

$$\delta_h(i) = (1-\gamma)^{i+1} + (\alpha\delta) \sum_{j=0}^i (1-\gamma)^{i-j} (1-\delta + \alpha\delta)^j.$$

Thus,

$$\begin{aligned} y_t - p_t &= -b_p + h_{t+1} \\ &= \left(-b_p + b_h'' \right) + \gamma a_t + \gamma \sum_{i=0}^{\infty} \delta_h(i) a_{t-1-i}. \end{aligned} \tag{63}$$

A.9 Proof of (24), (25)

Since the addition or subtraction of a constant term will not affect the stationarity property, in order to verify the stationarity of the house price-to-income ratio ($y_t - p_t$), it suffices to verify that the de-means h_{t+1} , i.e. \hat{h}_{t+1} ,

$$\hat{h}_{t+1} \equiv \gamma a_t + \gamma \sum_{i=0}^{\infty} \delta_h(i) a_{t-1-i}, \tag{64}$$

is stationary, where $\delta_h(i)$ is given by (44),

$$y_t - p_t = \hat{b}_{yp} + \hat{h}_{t+1},$$

with \hat{b}_{yp} being a constant.

Before we proceed, we observe that the term $\sum_{j=0}^i (1-\gamma)^{i-j} (1-\delta+\alpha\delta)^j$ is of the form $\sum_{i=0}^n a^{n-i} b^i$, which is well known to be equal to $\frac{a^{n+1}-b^{n+1}}{a-b}$. Thus, we have¹⁹

$$\begin{aligned} & \sum_{j=0}^i (1-\gamma)^{i-j} (1-\delta+\alpha\delta)^j \\ &= \frac{(1-\gamma)^{i+1} - (1-\delta+\alpha\delta)^{i+1}}{(1-\gamma) - (1-\delta+\alpha\delta)} \\ &= \frac{(1-\gamma)^{i+1} - (1-\delta+\alpha\delta)^{i+1}}{\delta - \gamma - \alpha\delta}, \end{aligned}$$

Hence, (44) can be further simplified,

$$\begin{aligned} \delta_h(i) &= (1-\gamma)^{i+1} + (\alpha\delta) \sum_{j=0}^i (1-\gamma)^{i-j} (1-\delta+\alpha\delta)^j \\ &= (1-\gamma)^{i+1} + (\alpha\delta) \frac{(1-\gamma)^{i+1} - (1-\delta+\alpha\delta)^{i+1}}{\delta - \gamma - \alpha\delta} \\ &= (1-\gamma)^{i+1} \left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right) + (1-\delta+\alpha\delta)^{i+1} \left(\frac{-\delta\alpha}{\delta - \gamma - \delta\alpha} \right). \end{aligned}$$

Thus, we can interpret $\delta_h(i)$ as a “weighted average” of two terms, $(1-\gamma)^{i+1}$ and $(1-\delta+\alpha\delta)^{i+1}$. Furthermore, since δ and α are strictly between 0 and 1, $0 < (1-\delta+\alpha\delta) = (1-\delta(1-\alpha)) < 1$. Therefore, $0 < (1-\delta+\alpha\delta)^{i+1} < (1-\delta+\alpha\delta)^i < 1$, $i = 0, 1, 2, \dots$. Similarly, since $0 < \gamma < 1$, $0 < (1-\gamma) < 1$. Therefore, we also have $0 < (1-\gamma)^{i+1} < (1-\gamma)^i < 1$. The following lemma summarizes our discussion,

Lemma 2 *Since $0 < \delta, \alpha, \gamma < 1$, we can express $\delta_h(i)$ as a weighted average of two functions which depend on i ,*

$$\delta_h(i) = \left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right) (1-\gamma)^{i+1} + \left(\frac{-\delta\alpha}{\delta - \gamma - \delta\alpha} \right) (1-\delta+\alpha\delta)^{i+1}, \quad (65)$$

where

$$\begin{aligned} 0 &< (1-\gamma)^{i+1}, \quad (1-\delta+\alpha\delta)^{i+1} < 1, \\ (1-\delta+\alpha\delta)^{i+1} &< (1-\delta+\alpha\delta)^i, \quad (1-\gamma)^{i+1} < (1-\gamma)^i. \end{aligned} \quad (66)$$

In fact, we can say more of the properties of the series $\{\delta_h(i)\}$. Notice from (65), it is of the form

$$\delta_h(i) = \varphi x^{i+1} + (1-\varphi) z^{i+1}, \quad (67)$$

¹⁹Notice that if the business capital and residential housing have the same depreciation rate, $\gamma \approx \delta$, then $\gamma - \delta(1-\alpha) \approx \gamma\alpha$.

where $0 < x, y < 1$, with the roles of x and y being symmetric, and the roles for φ and $(1 - \varphi)$ are symmetric as well. Thus, without loss of generality, we assume that

$$\begin{aligned}\varphi &= \left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right) \\ x &= (1 - \gamma) \\ z &= (1 - \delta + \alpha\delta).\end{aligned}\tag{68}$$

Notice that if $\delta > \gamma > \delta(1 - \alpha)$, then $\varphi < 0$. Thus, in general, the sign of φ is not known. Now, recall from equation (6) that

$$a_t = \rho a_{t-1} + \varepsilon_t,$$

where $a_t = \ln(A_t)$, $0 < \rho \leq 1$, and ε_t is a white noise process. Now, $0 < \rho < 1$,

$$a_t = \varepsilon_t + \sum_{j=1}^{\infty} \rho^j \varepsilon_{t-j}.\tag{69}$$

Substitute (69) into (64), we have

$$\widehat{h}_{t+1} = \gamma \varepsilon_t + \gamma \sum_{i=0}^{\infty} \delta'_h(i) \varepsilon_{t-1-i},$$

where

$$\delta'_h(i) = \sum_{j=0}^{i+1} \rho^{i+1-j} (\delta_h(j-1)),\tag{70}$$

with $\delta_h(-1) \equiv 1$. Clearly, it is in the form of (24).

By the Wold theorem, we need to check whether the following expression is finite,

$$\left| \gamma \left(1 + \sum_{i=0}^{\infty} \delta'_h(i) \right) \right|^2.$$

Notice that γ is a constant. Thus, $|\gamma (1 + \sum_{i=0}^{\infty} \delta'_h(i))|^2$ is finite if and only if $|\sum_{i=0}^{\infty} \delta'_h(i)|^2$ is finite.

By definition, i.e. (70), (67),

$$\begin{aligned}
\delta'_h(i) &= \sum_{j=0}^{i+1} \rho^{i+1-j} (\delta_h(j-1)) \\
&= \rho^{i+1} + \rho^i \cdot \delta_h(0) + \rho^{i-1} \cdot \delta_h(1) \\
&\quad + \dots + \rho \cdot \delta_h(i-1) + \delta_h(i) \\
&= \rho^{i+1} + \rho^i (\varphi x + (1-\varphi)z) + \rho^{i-1} (\varphi x^2 + (1-\varphi)z^2) \\
&\quad + \dots + \rho (\varphi x^i + (1-\varphi)z^i) + (\varphi x^{i+1} + (1-\varphi)z^{i+1}) \\
&= \varphi (\rho^{i+1} + \rho^i x + \rho^{i-1} x^2 + \dots + \rho x^i + x^{i+1}) \\
&\quad + (1-\varphi) (\rho^{i+1} + \rho^i z + \rho^{i-1} z^2 + \dots + \rho z^i + z^{i+1}) \\
&= \varphi \left(\frac{x^{i+2} - \rho^{i+2}}{x - \rho} \right) + (1-\varphi) \left(\frac{z^{i+2} - \rho^{i+2}}{z - \rho} \right) \\
&= \left(\frac{\varphi}{x - \rho} \right) x^{i+2} + \left(\frac{1-\varphi}{z - \rho} \right) z^{i+2} - \rho^{i+2} \left[\left(\frac{\varphi}{x - \rho} \right) + \left(\frac{1-\varphi}{z - \rho} \right) \right] \tag{71}
\end{aligned}$$

Equipped with this result, let us define both the infinite sum and the partial sum,

$$\begin{aligned}
S'(\infty) &\equiv \sum_{i=0}^{\infty} |\delta'_h(i)|^2, \\
S'(m) &\equiv \sum_{i=0}^m |\delta'_h(i)|^2.
\end{aligned}$$

It is well known that $S'(\infty)$ if and only if $S'(m)$ converges. And $S'(m)$ converges if it is a Cauchy sequence, meaning that $\forall \epsilon > 0, \exists N > 0$, such that $\forall n, m > N, |S'(m) - S'(n)| < \epsilon$ (See Bartle and Sherbert, 2011, among others, for more details). Without loss of generality, let us assume that $m > n$. By definition,

$$\begin{aligned}
&S'(m) - S'(n) \\
&= \sum_{i=n+1}^m \left| \sum_{j=0}^i \delta'_h(j) \right|^2.
\end{aligned}$$

Now define

$$\omega' = \max \left(\left| \frac{\varphi}{x - \rho} \right|, \left| \frac{1-\varphi}{z - \rho} \right| \right).$$

Since φ, x, z, ρ are all constant, ω' is a constant. Observe also that $0 < x, z, \rho < 1$. It implies that for any integer $i, 0 < x^i, z^i, \rho^i < 1, i = 1, 2, \dots$. Now define

$$\zeta = \max(x, z, \rho).$$

Clearly, ζ is still a constant and $0 < \zeta < 1$. Recall (71) that

$$\delta'_h(i) = \left(\frac{\varphi}{x-\rho}\right) x^{i+2} + \left(\frac{1-\varphi}{z-\rho}\right) z^{i+2} - \rho^{i+2} \left[\left(\frac{\varphi}{x-\rho}\right) + \left(\frac{1-\varphi}{z-\rho}\right) \right].$$

Following Hardy, Littlewood and Polya (1934), Steele (2004), we have

$$\begin{aligned} |\delta'_h(i)| &\leq \left| \frac{\varphi}{x-\rho} \right| x^{i+2} + \left| \frac{1-\varphi}{z-\rho} \right| z^{i+2} + \rho^{i+2} \left[\left| \frac{\varphi}{x-\rho} \right| + \left| \frac{1-\varphi}{z-\rho} \right| \right] \\ &< 4\omega'(\zeta)^{i+2}, \end{aligned}$$

which implies that

$$|\delta'_h(i)|^2 < 16 (\omega')^2 (\zeta)^{2i+4},$$

Thus,

$$\begin{aligned} &S'(m) - S'(n) \\ &= \sum_{i=n+1}^m |\delta'_h(i)|^2 \\ &< \sum_{i=n+1}^m 16 (\omega')^2 (\zeta)^{2i+4} \\ &= 16 (\omega')^2 (\zeta)^{2n+6} \left(\frac{1 - (\zeta)^{2(m-n-1)}}{1 - (\zeta)^2} \right) \\ &< \left(\frac{16 (\omega')^2}{1 - (\zeta)^2} \right) (\zeta)^{2n+6}. \end{aligned}$$

Notice that $0 < \zeta < 1$, $(\zeta)^{2n+6} < (\zeta)^{2N^\epsilon+6}$ if $n > N^\epsilon$. In addition, $\ln \zeta < 0$. Now define

$$B^{yp} \equiv \left(\frac{16 (\omega')^2}{1 - (\zeta)^2} \right).$$

It is then clear that $\forall \epsilon > 0, \forall m, n > N^\epsilon, |S'(m) - S'(n)| < B^{yp} (\zeta)^{2n+6} < B^{yp} (\zeta)^{2N^\epsilon+6} < \epsilon$, where

$$N^\epsilon > \frac{1}{2} \left\{ \frac{\ln \left[\frac{\epsilon}{B^{yp}} \right]}{\ln \zeta} - 6 \right\}.$$

In other words, $S'(\infty)$ converges and the log house price-to-income ratio, $(y_t - p_t)$, is stationary, which also proves (25).

B The case of persistent productivity shock, $\rho = 1$

In this appendix, we will study how the results in the text will be modified when the productivity shock is persistent, $\rho = 1$.

First, notice that results up to (14) are valid, whether the productivity shock is mean-reverting, $0 < \rho < 1$, or when the productivity shock is persistent, $\rho = 1$.

Second, some results need to be modified as the productivity shock becomes persistent. We will approach them in order.

- (15) needs to be modified as follows:

Proposition 9 *The stationarity of y_t depends on the persistency of the productivity shock a_t , which is defined in (6). In particular, when we express y_t in the form of $\sum_{j=0}^{\infty} \psi_j^{(y)} \varepsilon_{t-j}$, if $\rho = 1$, i.e. the productivity shock a_t is a permanent shock, then the income y_t is non-stationary,*

$$\sum_{j=0}^{\infty} \left| \psi_j^{(y)} \right|^2 = \infty. \quad (72)$$

Proof. By (6), we can re-write a_t as a function of previous period innovation terms, $\{\varepsilon_{t-j}\}_{j=0}$,

$$a_t = \sum_{j=0}^{\infty} \varepsilon_{t-j}, \quad (73)$$

and hence (14) can be written as

$$y_t = b_y'' + \varepsilon_t + \sum_{i=1}^{\infty} \left(1 + (\alpha\delta) \sum_{j=0}^{i-1} (1 - \delta + \alpha\delta)^j \right) \varepsilon_{t-i}. \quad (74)$$

By Wold's Decomposition Theorem, to verify the stationarity of the income process y_t , it suffices to check whether

$$\sum_{i=1}^{\infty} \left| 1 + (\alpha\delta) \sum_{j=0}^{i-1} (1 - \delta + \alpha\delta)^j \right|^2 < \infty.$$

Clearly,

$$\begin{aligned} & \sum_{i=1}^{\infty} \left| 1 + (\alpha\delta) \sum_{j=0}^{i-1} (1 - \delta + \alpha\delta)^j \right|^2 \\ & > \sum_{i=1}^{\infty} |1|^2 \text{ as } (\alpha\delta), (1 - \delta + \alpha\delta) > 0 \\ & = \infty. \end{aligned}$$

Thus, the income process is non-stationary, and it proves (72). ■

- And while (16) is still valid, (17) needs to be modified.

Proposition 10 *The stationarity of p_t depends on the persistency of the productivity shock a_t , which is defined in (6). In particular, when we express p_t in the form of $\sum_{j=0}^{\infty} \psi_j^{(p)} \varepsilon_{t-j}$, and if $\rho = 1$, i.e. the productivity shock a_t is persistent, then the house price p_t is non-stationary,*

$$\sum_{j=0}^{\infty} \left| \psi_j^{(p)} \right|^2 < \infty. \quad (75)$$

Proof. The proof here is similar to the proof of (17). As in the case when the productivity shock is mean-reverting, i.e. $0 < \rho < 1$, we need to first obtain an expression of the house price in terms of the productivity shocks $\{a_t\}$. The proof of (17) shows that (58),

$$\begin{aligned} p_t &= b_p'' + (1 - \gamma) a_t + (1 - \gamma) [\alpha\delta - \gamma] a_{t-1} \\ &\quad + (1 - \gamma) [(\alpha\delta)(1 - \delta + \alpha\delta) - \gamma(\alpha\delta) - \gamma(1 - \gamma)] a_{t-2} \\ &\quad + \sum_{i=3}^{\infty} \delta^p(i) a_{t-i}, \end{aligned}$$

with (54),

$$\text{where } \delta^p(i) \equiv (1 - \gamma) \left[(\alpha\delta) \left(\frac{\delta - \alpha\delta}{\delta - \alpha\delta - \gamma} \right) (1 - \delta + \alpha\delta)^{i-1} - \gamma \left(\frac{\delta - \gamma}{\delta - \alpha\delta - \gamma} \right) (1 - \gamma)^{i-1} \right].$$

We further observe that if we use the following notations, (55),

$$\begin{aligned} x &= (1 - \gamma) \\ z &= (1 - \delta + \alpha\delta) \\ \mathcal{A} &= (1 - \gamma) (\alpha\delta) \left(\frac{\delta - \alpha\delta}{\delta - \alpha\delta - \gamma} \right) \\ \mathcal{B} &= (1 - \gamma) \gamma \left(\frac{\delta - \gamma}{\delta - \alpha\delta - \gamma} \right), \end{aligned}$$

then we have (56) and (57),

$$\begin{aligned} 0 &< x, z < 1, \\ \delta^p(i) &= \mathcal{A}(z)^{i-1} - \mathcal{B}(x)^{i-1}. \end{aligned}$$

Hence, to verify the stationarity of p_t , it suffices to check whether $\sum_{i=3}^{\infty} \delta^p(i) a_{t-i}$ is stationary or not. Again, recall from equation (6) that

$$a_t = \rho a_{t-1} + \varepsilon_t,$$

where $a_t = \ln(A_t)$, $0 < \rho \leq 1$, and ε_t is a white noise process. Now, with $\rho = 1$, we have (73), and hence

$$\begin{aligned} & \sum_{i=3}^{\infty} \delta^p(i) a_{t-i} \\ &= \sum_{i=3}^{\infty} \left(\sum_{j=3}^i \delta^p(j) \right) \varepsilon_{t-i}. \end{aligned}$$

By the Wold's decomposition theorem, p_t is stationary if $\sum_{i=3}^{\infty} \left| \sum_{j=3}^i \delta^p(j) \right|^2 < \infty$. It is well known that $\sum_{i=3}^{\infty} \left| \sum_{j=3}^i \delta^p(j) \right|^2 < \infty$ if and only if $\forall \epsilon > 0, \exists N > 0$, such that $\forall n > m > N, \sum_{i=m}^n \left| \sum_{j=3}^i \delta^p(j) \right|^2 < \epsilon$ (see Bartle and Sherbert, 2011). By (57), (55),

$$\begin{aligned} & \sum_{j=3}^i \delta^p(j) \\ &= \mathcal{A} \left(z^2 \frac{1-z^{i-2}}{1-z} \right) - \mathcal{B} \left(x^2 \frac{1-x^{i-2}}{1-x} \right). \end{aligned}$$

Recall that (for instance, see Hardy, Littlewood and Polya, 1934; Steele, 2004)

$$x^2 + y^2 - 2|x||y| \leq |x-y|^2 \leq x^2 + y^2 + 2|x||y|.$$

Therefore,

$$\begin{aligned} & \left| \sum_{j=3}^i \delta^p(j) \right|^2 \\ &= \left| \mathcal{A} \left(z^2 \frac{1-z^{i-2}}{1-z} \right) - \mathcal{B} \left(x^2 \frac{1-x^{i-2}}{1-x} \right) \right|^2 \\ &\geq \left| \mathcal{A} \left(z^2 \frac{1-z^{i-2}}{1-z} \right) \right|^2 + \left| \mathcal{B} \left(x^2 \frac{1-x^{i-2}}{1-x} \right) \right|^2 \\ &\quad - 2 \left| \mathcal{A} \left(z^2 \frac{1-z^{i-2}}{1-z} \right) \right| \left| \mathcal{B} \left(x^2 \frac{1-x^{i-2}}{1-x} \right) \right| \\ &= |\mathcal{A}|^2 \left| z^2 \frac{1-z^{i-2}}{1-z} \right|^2 + |\mathcal{B}|^2 \left| x^2 \frac{1-x^{i-2}}{1-x} \right|^2 \\ &\quad - 2|\mathcal{A}||\mathcal{B}| \left(z^2 \frac{1-z^{i-2}}{1-z} \right) \left(x^2 \frac{1-x^{i-2}}{1-x} \right), \end{aligned}$$

as (56) implies that $\left(z^2 \frac{1-z^{i-2}}{1-z} \right), \left(x^2 \frac{1-x^{i-2}}{1-x} \right) > 0$. In fact, due to (56), $0 < x^i, z^i < 1$,

$\forall i \geq 3$. Thus,

$$\begin{aligned}
& \left| \sum_{j=3}^i \delta^p(j) \right|^2 \\
& \geq |\mathcal{A}|^2 \left| z^2 \frac{1-z^{i-2}}{1-z} \right|^2 + |\mathcal{B}|^2 \left| x^2 \frac{1-x^{i-2}}{1-x} \right|^2 - 2|\mathcal{A}||\mathcal{B}| \left(z^2 \frac{1-z^{i-2}}{1-z} \right) \left(x^2 \frac{1-x^{i-2}}{1-x} \right) \\
& = |\mathcal{A}|^2 \left| \frac{z^2 - z^i}{1-z} \right|^2 + |\mathcal{B}|^2 \left| \frac{x^2 - x^i}{1-x} \right|^2 - 2|\mathcal{A}||\mathcal{B}| \left(\frac{z^2 - z^i}{1-z} \right) \left(\frac{x^2 - x^i}{1-x} \right) \equiv \mathcal{A}_{pi}.
\end{aligned}$$

Notice that $\left| z^2 \frac{1-z^{i-2}}{1-z} \right|^2 = \left| \frac{z^2 - z^i}{1-z} \right|^2$. Notice also that as $i \rightarrow \infty$, x^i and z^i both converges to zero by (56). Thus, $\mathcal{A}_{pi} \rightarrow \mathcal{A}_p$ as $i \rightarrow \infty$, where

$$\begin{aligned}
\mathcal{A}_p & \equiv |\mathcal{A}|^2 \left\{ \left(\frac{z^2}{1-z} \right)^2 \right\} + |\mathcal{B}|^2 \left\{ \left(\frac{x^2}{1-x} \right)^2 \right\} - 2|\mathcal{A}||\mathcal{B}| \left(\frac{z^2}{1-z} \right) \left(\frac{x^2}{1-x} \right) \\
& = \left\{ \mathcal{A} \left(\frac{z^2}{1-z} \right) - \mathcal{B} \left(\frac{x^2}{1-x} \right) \right\}^2.
\end{aligned}$$

Notice that by (55), $\mathcal{A} \left(\frac{z^2}{1-z} \right) = (1-\gamma)(\alpha\delta)(1-\delta+\alpha\delta)^2(\delta-\alpha\delta-\gamma)^{-1}$, and $\mathcal{B} \left(\frac{x^2}{1-x} \right) = (1-\gamma)^3 \left(\frac{\delta-\gamma}{\delta-\alpha\delta-\gamma} \right)$. Thus,

$$\mathcal{A}_p = \left\{ \frac{(1-\gamma)}{(\delta-\alpha\delta-\gamma)} \left[(\alpha\delta)(1-\delta+\alpha\delta)^2 - (1-\gamma)^2(\delta-\gamma) \right] \right\}^2$$

which is positive except when the combination of the parameters is such that $(\alpha\delta)(1-\delta+\alpha\delta)^2 = (1-\gamma)^2(\delta-\gamma)$. Therefore, with all these results, $\exists N > 0$, such that for all $n > m > N$, $|\mathcal{A}_{pn} - \mathcal{A}_p|, |\mathcal{A}_{pm} - \mathcal{A}_p| < \epsilon$. Thus, $\sum_{i=m}^n \left| \sum_{j=3}^i \delta^p(j) \right|^2 > \sum_{i=m}^n (\mathcal{A}_p - \epsilon) = (n-m)(\mathcal{A}_p - \epsilon) > 0$. Thus, $\sum_{i=3}^{\infty} \left| \sum_{j=3}^i \delta^p(j) \right|^2 \not\leq \infty$ and p_t is not stationary. This completes the proof of (75). ■

- Most of the main results are valid. However, (25) needs to be modified:

Proposition 11 *Assume that (6) holds. We can then write the house price-to-income ratio in the following form,*

$$(y_t - p_t) = \widehat{b}_{yp} + \sum_{i=0}^{\infty} \delta'_h(i) \varepsilon_{t-i},$$

where ε_t is a white noise. Furthermore, if $\rho = 1$, i.e. the productivity shock a_t is a permanent shock, the stochastic process of log house price-to-income ratio is not stationary,

$$\sum_{j=0}^{\infty} |\delta'_h(i)|^2 = \infty, \quad (76)$$

Proof. In the proof (25), we already verify that

$$(y_t - p_t) = \widehat{b}_{yp} + \sum_{i=0}^{\infty} \delta'_h(i) \varepsilon_{t-i},$$

where

$$\delta_h(i) = \left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right) (1 - \gamma)^{i+1} + \left(\frac{-\delta\alpha}{\delta - \gamma - \delta\alpha} \right) (1 - \delta + \alpha\delta)^{i+1},$$

and

$$\begin{aligned} 0 &< (1 - \gamma)^{i+1}, \quad (1 - \delta + \alpha\delta)^{i+1} < 1, \\ (1 - \delta + \alpha\delta)^{i+1} &< (1 - \delta + \alpha\delta)^i, \quad (1 - \gamma)^{i+1} < (1 - \gamma)^i. \end{aligned}$$

Recall that

$$\begin{aligned} \widehat{h}_{t+1} &= (y_t - p_t) - \widehat{b}_{yp} \\ &\equiv \gamma a_t + \gamma \sum_{i=0}^{\infty} \delta_h(i) a_{t-1-i}, \end{aligned}$$

where \widehat{b}_{yp} being a constant stationary, where $\delta_h(i)$ is given by (44). When $\rho = 1$, we have $a_t = \sum_{j=0}^{\infty} \varepsilon_{t-j}$, and when we substitute this into this expression,

$$\begin{aligned} \widehat{h}_{t+1} &\equiv \gamma \left(\sum_{j=0}^{\infty} \varepsilon_{t-j} \right) + \gamma \sum_{i=0}^{\infty} \delta_h(i) \left(\sum_{j=0}^{\infty} \varepsilon_{t-1-i-j} \right) \\ &= \gamma \left[\sum_{i=0}^{\infty} \left(\sum_{j=0}^i \delta_h(j-1) \right) \varepsilon_{t-i} \right], \end{aligned}$$

with $\delta_h(-1) \equiv 1$. Hence

$$\begin{aligned} y_t - p_t &= \widehat{b}_{yp} + \widehat{h}_{t+1} \\ &= \widehat{b}_{yp} + \gamma \left[\sum_{i=0}^{\infty} \left(\sum_{j=0}^i \delta_h(j-1) \right) \varepsilon_{t-i} \right], \end{aligned}$$

which is of the form of (24). Since ε_t is white noise, \widehat{h}_{t+1} is stationary if (46) holds, by the Wold theorem. It means that we need to check whether the following expression is finite,

$$\sum_{i=0}^{\infty} \left| \sum_{j=0}^i \delta_h(j-1) \right|^2.$$

For future reference, let us define both the infinite sum and the partial sum,

$$S(\infty) \equiv \sum_{i=0}^{\infty} \left| \sum_{j=0}^i \delta_h(j-1) \right|^2,$$

$$S(m) \equiv \sum_{i=0}^m \left| \sum_{j=0}^i \delta_h(j-1) \right|^2.$$

It is well known that $S(\infty)$ if and only if $S(m)$ converges. And $S(m)$ converges if it is a Cauchy sequence, meaning that $\forall \epsilon > 0, \exists N > 0$, such that $\forall n, m > N, |S(m) - S(n)| < \epsilon$ (See Bartle and Sherbert, 2011, among others, for more details). Without loss of generality, let us assume that $m > n$. By definition,

$$S(m) - S(n) = \sum_{i=n+1}^m \left| \sum_{j=0}^i \delta_h(j-1) \right|^2.$$

By (65), (67), for $i = (n+1), \dots, m$,

$$\begin{aligned} & \sum_{j=0}^i \delta_h(j-1) \\ &= \delta_h(-1) + \delta_h(0) + \dots + \delta_h(i-1) \\ &= 1 + \varphi x + (1-\varphi)z + \dots \\ & \quad + \varphi x^i + (1-\varphi)z^i \\ &= 1 + \varphi x \frac{1-x^i}{1-x} + (1-\varphi)z \frac{1-z^i}{1-z}. \end{aligned} \tag{77}$$

Notice that $0 < x, z < 1$, and hence $0 < x^i < x < 1, 0 < z^i < z < 1$, where x, z are $(1-\gamma), (1-\delta + \alpha\delta)$. Also, recall that $|a+b+c| \geq |a|^2 + |b|^2 + |c|^2 - 2|a||b| - 2|a||c| - 2|b||c|$, (for instance, see Hardy, Littlewood and Polya, 1934; Steele, 2004). Thus,

$$\begin{aligned} & \left| \sum_{j=0}^i \delta_h(j-1) \right|^2 \\ &= \left| 1 + \varphi x \frac{1-x^i}{1-x} + (1-\varphi)z \frac{1-z^i}{1-z} \right|^2 \equiv \mathcal{A}_{yi}. \end{aligned}$$

Since $0 < x, z < 1$, $\mathcal{A}_{yi} \rightarrow \mathcal{A}_y$ as $i \rightarrow \infty$, where

$$\mathcal{A}_y \equiv \left| 1 + \varphi x \frac{1}{1-x} + (1-\varphi) z \frac{1}{1-z} \right|^2.$$

Clearly, \mathcal{A}_y is independent of i and by (68), $\varphi x \frac{1}{1-x} = \left(\frac{\delta-\gamma}{\delta-\gamma-\delta\alpha} \right) \left(\frac{1-\gamma}{\gamma} \right)$ and $(1-\varphi) z \frac{1}{1-z} = \left(\frac{-\delta\alpha}{\delta-\gamma-\delta\alpha} \right) \left(\frac{1-\delta+\alpha\delta}{\delta-\alpha\delta} \right)$, and hence

$$\mathcal{A}_y \equiv \left| \frac{1}{\gamma(1-\alpha)} \right|^2 > 0.$$

Thus, $\forall \epsilon > 0$, there exists $N > 0$, such that $\forall m, n > N$, $|\mathcal{A}_{yi} - \mathcal{A}_y| < \epsilon$. And

$$\begin{aligned} & S(m) - S(n) \\ &= \sum_{i=n+1}^m \left| \sum_{j=0}^i \delta_h(j-1) \right|^2 \\ &= \sum_{i=n+1}^m \mathcal{A}_{yi} \\ &> \sum_{i=n+1}^m (\mathcal{A}_y - \epsilon) \\ &= (m-n)(\mathcal{A}_y - \epsilon) > 0, \end{aligned}$$

which is *not* converging. In other words, $S(\infty)$ does not converge, which proves (76). ■

- In the main text, we have proved the cointegration of output y_t and house price p_t when the productivity shock is mean-reverting, $0 < \rho < 1$. Now we will show that output and house price are cointegrated even when the productivity shock is permanent, i.e. $\rho = 1$.

Proposition 12 *Assume that (6) holds and $\rho = 1$. There exists a constant θ^* , such that $(y_t - \theta^* p_t)$ is stationary. It means that if $(y_t - \theta^* p_t) = \sum_{i=0}^{\infty} \alpha_\theta(i) \varepsilon_{t-i}$, where ε_t is a white noise, then*

$$\sum_{i=0}^{\infty} |\alpha_\theta(i)|^2 < \infty. \quad (78)$$

Proof. To verify whether y_t and p_t are cointegrated, we need to find a constant θ , at which the expression $(y_t - \theta p_t)$ is stationary. Thus, we need to obtain an expression of $(y_t - \theta p_t)$ based on some previous results. Recall (14) that

$$h_t = b_h'' + \gamma a_{t-1} + \gamma \sum_{i=0}^{\infty} \delta_h(i) a_{t-2-i},$$

and (16) that

$$p_t = b_p + y_t - h_{t+1}.$$

Together, they yield

$$p_t = b_p + y_t - \left(b_h'' + \gamma a_t + \gamma \sum_{i=0}^{\infty} \delta_h(i) a_{t-1-i} \right).$$

Thus, $(y_t - \theta p_t)$, $\theta > 0$, can be expressed as

$$\begin{aligned} & y_t - \theta p_t \\ &= b_{yp} + (1 - \theta) y_t + \theta \gamma \left(a_t + \sum_{i=0}^{\infty} \delta_h(i) a_{t-1-i} \right). \end{aligned} \quad (79)$$

where b_{yp} is a constant term, $b_{yp} \equiv -\theta (b_p - b_h'')$. By (41), we have

$$y_t = b_y'' + a_t + (\alpha \delta) \sum_{i=0}^{\infty} (1 - \delta + \alpha \delta)^i a_{t-1-i}.$$

Combining (79) and (41), we have

$$\begin{aligned} & y_t - \theta p_t \\ &= b'_{yp} \\ & \quad + (1 - \theta) \left[a_t + (\alpha \delta) \sum_{i=0}^{\infty} (1 - \delta + \alpha \delta)^i a_{t-1-i} \right] \\ & \quad + \theta \gamma \left(a_t + \sum_{i=0}^{\infty} \delta_h(i) a_{t-1-i} \right) \\ &= b'_{yp} + [(1 - \theta) + \theta \gamma] a_t \\ & \quad + \sum_{i=0}^{\infty} \delta_{\theta}(i) a_{t-1-i}, \end{aligned} \quad (80)$$

where b'_{yp} is a constant, $b'_{yp} \equiv (1 - \theta) b_y'' + b_{yp}$, and

$$\delta_{\theta}(i) = (1 - \theta) (\alpha \delta) (1 - \delta + \alpha \delta)^i + \theta \gamma \delta_h(i). \quad (81)$$

■

Remark 1 If $\theta = 1$, (80) is equivalent to (23).

Proof. If $\theta = 1$, $[(1 - \theta) + \theta\gamma] = \gamma$, $\delta_\theta(i) = (1 - 1)(\alpha\delta)(1 - \delta + \alpha\delta)^i + 1 \cdot \gamma\delta_h(i) = \gamma\delta_h(i)$, and hence (80) is reduced to (23).

Since we already know that if $0 < \rho < 1$, $(y_t - p_t)$ is stationary (i.e. $\theta = 1$). Now we focus on the case $\rho = 1$. In that case, by (73), (80) can be re-written as

$$\begin{aligned} y_t - \theta p_t &= b'_{yp} + [(1 - \theta) + \theta\gamma] a_t + \sum_{i=0}^{\infty} \delta_\theta(i) a_{t-1-i} \\ &= b'_{yp} + [(1 - \theta) + \theta\gamma] \sum_{j=0}^{\infty} \varepsilon_{t-j} + \sum_{i=0}^{\infty} \delta_\theta(i) \sum_{j=0}^{\infty} \varepsilon_{t-1-i-j} \\ &= b'_{yp} + [(1 - \theta) + \theta\gamma] \varepsilon_t \\ &\quad + \sum_{i=1}^{\infty} \left[[(1 - \theta) + \theta\gamma] + \sum_{j=1}^i \delta_\theta(j - 1) \right] \varepsilon_{t-i}, \end{aligned}$$

where $\sum_{j=1}^1 \delta_\theta(j - 1) \equiv 1$. By Wold's Decomposition Theorem, $\{y_t - \theta p_t\}$ is stationary if $\sum_{i=1}^{\infty} \left| [(1 - \theta) + \theta\gamma] + \sum_{j=1}^i \delta_\theta(j - 1) \right|^2 + [(1 - \theta) + \theta\gamma]^2$ is finite. Since $[(1 - \theta) + \theta\gamma]^2$ is a constant and finite, the condition will translate into "if $\sum_{i=1}^{\infty} \left| [(1 - \theta) + \theta\gamma] + \sum_{j=1}^i \delta_\theta(j - 1) \right|^2$ is finite." Following the same procedure of the previous proof, we define both the infinite sum and the partial sum,

$$\begin{aligned} S_\theta(\infty) &\equiv \sum_{i=1}^{\infty} \left| [(1 - \theta) + \theta\gamma] + \sum_{j=1}^i \delta_\theta(j - 1) \right|^2, \\ S_\theta(m) &\equiv \sum_{i=1}^m \left| [(1 - \theta) + \theta\gamma] + \sum_{j=1}^i \delta_\theta(j - 1) \right|^2. \end{aligned}$$

It is well known that $S_\theta(\infty)$ is finite if and only if $S_\theta(m)$ converges. And $S_\theta(m)$ converges if it is a Cauchy sequence, meaning that $\forall \epsilon > 0$, $\exists N > 0$, such that $\forall n, m > N$, $|S_\theta(m) - S_\theta(n)| < \epsilon$ (See Bartle and Sherbert, 2011, among others, for more details). Without loss of generality, let us assume that $m > n$. By definition,

$$\begin{aligned} &S_\theta(m) - S_\theta(n) \\ &= \sum_{i=n+1}^m \left| [(1 - \theta) + \theta\gamma] + \sum_{j=1}^i \delta_\theta(j - 1) \right|^2. \end{aligned} \tag{82}$$

Recall (67), (68) that

$$\delta_h(i) = \varphi x^{i+1} + (1 - \varphi) z^{i+1},$$

where

$$\begin{aligned}\varphi &= \left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right) \\ x &= (1 - \gamma) > 0 \\ z &= (1 - \delta + \alpha\delta) > 0.\end{aligned}$$

Thus, (81) can be re-written as

$$\begin{aligned}\delta_\theta(i) &= (1 - \theta)(\alpha\delta)(1 - \delta + \alpha\delta)^i + \theta\gamma\delta_h(i) \\ &= (1 - \theta)(\alpha\delta)z^i + \theta\gamma[\varphi x^{i+1} + (1 - \varphi)z^{i+1}],\end{aligned}$$

and we can re-write (82),

$$S_\theta(m) - S_\theta(n) = \sum_{i=n+1}^m |S_{\theta i}|^2 \quad (83)$$

where $S_{\theta i}$ is defined as the following,

$$\begin{aligned}S_{\theta i} &\equiv [(1 - \theta) + \theta\gamma] + \sum_{j=1}^i \delta_\theta(j - 1) \\ &= [(1 - \theta) + \theta\gamma] \\ &\quad + (1 - \theta)(\alpha\delta) + \theta\gamma[\varphi x + (1 - \varphi)z] \\ &\quad + (1 - \theta)(\alpha\delta)z + \theta\gamma[\varphi x^2 + (1 - \varphi)z^2] \\ &\quad + \dots \\ &\quad + (1 - \theta)(\alpha\delta)z^{i-1} + \theta\gamma[\varphi x^i + (1 - \varphi)z^i] \\ &= [(1 - \theta) + \theta\gamma] + (1 - \theta)(\alpha\delta) \frac{1 - z^i}{1 - z} \\ &\quad + \theta\gamma \left[\varphi x \frac{1 - x^i}{1 - x} + (1 - \varphi)z \frac{1 - z^i}{1 - z} \right] \\ &= [(1 - \theta) + \theta\gamma] + \theta\gamma \left[\varphi x \frac{1 - x^i}{1 - x} \right] \\ &\quad + [(1 - \theta)(\alpha\delta) + \theta\gamma(1 - \varphi)z] \left(\frac{1 - z^i}{1 - z} \right).\end{aligned}$$

Notice that $S_{\theta i}$ can be re-written as

$$\begin{aligned}
S_{\theta i} &= [(1 - \theta) + \theta\gamma] + \theta\gamma \left[\varphi x \frac{1 - x^i}{1 - x} \right] \\
&\quad + [(1 - \theta)(\alpha\delta) + \theta\gamma(1 - \varphi)z] \left(\frac{1 - z^i}{1 - z} \right) \\
&= [(1 - \theta) + \theta\gamma] + \theta\gamma \left[\varphi x \frac{1}{1 - x} \right] \\
&\quad + [(1 - \theta)(\alpha\delta) + \theta\gamma(1 - \varphi)z] \left(\frac{1}{1 - z} \right) \\
&\quad - \left\{ \theta\gamma \left[\varphi x \frac{x^i}{1 - x} \right] \right. \\
&\quad \left. + [(1 - \theta)(\alpha\delta) + \theta\gamma(1 - \varphi)z] \left(\frac{z^i}{1 - z} \right) \right\}. \tag{84}
\end{aligned}$$

Intuitively, if

$$[(1 - \theta) + \theta\gamma] + \theta\gamma \left[\varphi x \frac{1}{1 - x} \right] + [(1 - \theta)(\alpha\delta) + \theta\gamma(1 - \varphi)z] \left(\frac{1}{1 - z} \right) = 0, \tag{85}$$

then $S_{\theta i}$ will be significantly simplified,

$$S_{\theta i} = - \left\{ \theta\gamma \left[\varphi x \frac{x^i}{1 - x} \right] + [(1 - \theta)(\alpha\delta) + \theta\gamma(1 - \varphi)z] \left(\frac{z^i}{1 - z} \right) \right\}$$

and since $0 < x, z < 1$, $x^i, z^i \rightarrow 0$ as $i \rightarrow \infty$, and hence $S_{\theta i} \rightarrow 0$. Thus, we need to choose θ such that (85) holds. Clearly, (85) holds if and only if

$$\left(1 + \frac{(\alpha\delta)}{1 - z} \right) = \theta \left\{ \left(1 + \frac{(\alpha\delta)}{1 - z} \right) - \gamma \left[(1 - \varphi) \frac{z}{1 - z} + \varphi \frac{x}{1 - x} \right] \right\}. \tag{86}$$

By (68),

$$\begin{aligned}
\frac{x}{1 - x} &= \frac{1 - \gamma}{\gamma} \\
\frac{1}{1 - z} &= \frac{1}{\delta(1 - \alpha)} \\
\frac{z}{1 - z} &= \frac{1 - \delta(1 - \alpha)}{\delta(1 - \alpha)} \\
\varphi &= \left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right),
\end{aligned}$$

which means that (86) can be further simplified as

$$\begin{aligned}
\frac{1}{1 - \alpha} &= \theta \left\{ \left(\frac{1}{1 - \alpha} \right) - \gamma \left[\left(\frac{-\alpha}{1 - \alpha} \right) \left(\frac{1 - \delta(1 - \alpha)}{\delta - \gamma - \delta\alpha} \right) + \left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right) \left(\frac{1 - \gamma}{\gamma} \right) \right] \right\} \\
&= \theta \left\{ \frac{\alpha^2\delta + \delta + \alpha\gamma - \gamma - 2\alpha\delta}{(1 - \alpha)(\delta - \gamma - \delta\alpha)} \right\}. \tag{87}
\end{aligned}$$

Since $0 < \alpha < 1$, (87) implies that

$$\begin{aligned}\theta &= \frac{(\delta - \gamma - \delta\alpha)}{\alpha^2\delta + \alpha\gamma - \alpha\delta + (\delta - \gamma - \alpha\delta)} \\ &= \frac{1}{1 - \alpha}.\end{aligned}\tag{88}$$

As $0 < \alpha < 1$, $\theta > 1$. Also, since $\theta = \frac{1}{1-\alpha}$, $1 - \theta = \frac{-\alpha}{1-\alpha} = -\alpha\theta$.

Observe further that as (85) holds, (84) can be simplified as

$$\begin{aligned}S_{\theta i} &= -\left\{\theta\gamma\left[\varphi x\frac{x^i}{1-x}\right] + [(1-\theta)(\alpha\delta) + \theta\gamma(1-\varphi)z]\left(\frac{z^i}{1-z}\right)\right\} \\ &= \mathbb{D}_1x^i + \mathbb{D}_2z^i,\end{aligned}\tag{89}$$

where

$$\begin{aligned}\mathbb{D}_1 &\equiv -\theta\gamma\varphi\frac{x}{1-x} \\ &= \left(\frac{1-\gamma}{1-\alpha}\right)\left(\frac{\delta-\gamma}{\delta-\gamma-\delta\alpha}\right), \\ \mathbb{D}_2 &\equiv -[(1-\theta)(\alpha\delta) + \theta\gamma(1-\varphi)z]\frac{1}{1-z} \\ &= \frac{-\alpha}{(1-\alpha)^2}\left[\alpha + \gamma\left(\frac{1-\delta+\alpha\delta}{\delta-\gamma-\alpha\delta}\right)\right],\end{aligned}$$

and hence both \mathbb{D}_1 and \mathbb{D}_2 are constant. For future reference, let us define

$$\nu_x = \max\{x, z\}.$$

Since $0 < x, z < 1$, $0 < \nu_x < 1$. Hence,

$$0 < (\nu_x)^i < (\nu_x)^j < \nu_x < 1, \forall i > j > 1.\tag{90}$$

Similarly, define

$$\nu_D = \max\{|\mathbb{D}_1|, |\mathbb{D}_2|\}.$$

Now combine (89) with (83), and follow Hardy, Littlewood and Polya (1934), Steele (2004),

we have

$$\begin{aligned}
& S_\theta(m) - S_\theta(n) \\
&= \sum_{i=n+1}^m |S_{\theta i}|^2 \\
&= \sum_{i=n+1}^m |\mathbb{D}_1 x^i + \mathbb{D}_2 z^i|^2 \\
&= \sum_{i=n+1}^m \left| (\mathbb{D}_1 x^i)^2 + (\mathbb{D}_2 z^i)^2 + 2(\mathbb{D}_1 x^i)(\mathbb{D}_2 z^i) \right| \\
&< \sum_{i=n+1}^m \left\{ |\nu_D x^i|^2 + |\nu_D z^i|^2 + 2(x^i)(z^i)|\mathbb{D}_1||\mathbb{D}_2| \right\} \\
&< |\nu_D|^2 \sum_{i=n+1}^m |x^i + z^i|^2 \\
&\leq |\nu_D|^2 \sum_{i=n+1}^m \left| 2(\nu_x)^i \right|^2 \\
&= A_\theta \sum_{i=n+1}^m (\nu_x)^{2i},
\end{aligned}$$

where

$$A_\theta \equiv 4|\nu_D|^2.$$

Therefore, $\forall \epsilon > 0, \forall m, n > N^\epsilon$,

$$\begin{aligned}
& S_\theta(m) - S_\theta(n) \\
&\leq A_\theta \sum_{i=n+1}^m (\nu_x)^{2i} \\
&< A_\theta \sum_{i=n+1}^{\infty} (\nu_x)^{2i} \text{ as } 0 < \nu_x < 1 \\
&= A_\theta \left(\frac{1}{1 - (\nu_x)^2} \right) (\nu_x)^{2(n+1)} \\
&< A_\theta \left(\frac{1}{1 - (\nu_x)^2} \right) (\nu_x)^{2(N^\epsilon+1)} \text{ as } 0 < \nu_x < 1.
\end{aligned}$$

Now define

$$B^{y\theta} \equiv A_\theta \left(\frac{1}{1 - (\nu_x)^2} \right).$$

And since $0 < \nu_x < 1$, $\ln \nu_x < 0$. Thus, $|S_\theta(m) - S_\theta(n)| < B^{y\theta} (\nu_x)^{2(N^\epsilon+1)} < \epsilon$,

$$\text{if } N^\epsilon > \frac{1}{2} \left(\frac{\ln(\epsilon/B^{y\theta})}{\ln \nu_x} \right) - 1.$$

Thus, $S_\theta(m)$ converges and hence $(y_t - \theta p_t)$ is stationary. ■

C More Discussion on δ and γ

Throughout the paper, the relative magnitude of two parameters, δ and γ , are of great importance. For instance, in the previous proof,

$$\begin{aligned} \delta_h(i) &= (1-\gamma)^{i+1} + (\alpha\delta) \sum_{j=0}^i (1-\gamma)^{i-j} (1-\delta + \alpha\delta)^j \\ &= (1-\gamma)^{i+1} + (\alpha\delta) \frac{(1-\gamma)^{i+1} - (1-\delta + \alpha\delta)^{i+1}}{\delta - \gamma - \alpha\delta} \\ &= (1-\gamma)^{i+1} \left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right) + (1-\delta + \alpha\delta)^{i+1} \left(\frac{-\delta\alpha}{\delta - \gamma - \delta\alpha} \right). \end{aligned}$$

Thus, we can interpret $\delta_h(i)$ as a “weighted average” of two terms, $(1-\gamma)^{i+1}$ and $(1-\delta + \alpha\delta)^{i+1}$. Furthermore, since δ and α are strictly between 0 and 1, $0 < (1-\delta + \alpha\delta) = (1-\delta(1-\alpha)) < 1$. Therefore, $0 < (1-\delta + \alpha\delta)^{i+1} < (1-\delta + \alpha\delta)^i < 1$, $i = 0, 1, 2, \dots$. Similarly, since $0 < \gamma < 1$, $0 < (1-\gamma) < 1$. Therefore, we also have $0 < (1-\gamma)^{i+1} < (1-\gamma)^i < 1$.

How about the “weights,” i.e. $\left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right)$ and $\left(\frac{-\delta\alpha}{\delta - \gamma - \delta\alpha} \right)$? Clearly, if $(\delta - \gamma - \delta\alpha) > 0$, $\left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha} \right) > 1$ and $\left(\frac{-\delta\alpha}{\delta - \gamma - \delta\alpha} \right) < 0$. In the macroeconomic literature, the capital share of output, α , has been estimated many times and it is typically around 0.3 to 0.35 (Cooley, 1995). For simplicity, let us assume that $\alpha = 1/3$. On the other hand, empirical estimates for the parameters δ and γ are *quite different* that it may affect how we proceed. Let us take a few examples to illustrate.

1. According to Greenwood and Hercowitz (1991), $\delta = \gamma = 0.078$ on quarterly basis. In that case, $\delta - \gamma = 0$, and hence

$$\delta_h(i) = (1-\delta + \alpha\delta)^{i+1},$$

which is clearly a converging sequence as $0 < (1-\delta + \alpha\delta) < 1$.

2. According to Davis and Heathcote (2005), $\delta = 0.056$, $\gamma = 0.016$ on quarterly basis. In that case, $\delta - \gamma = 0.04$, $\delta - \gamma - \delta\alpha = 0.056 - 0.016 - (0.056) * (1/3) = 0.021 > 0$. Hence

$$\delta_h(i) \approx 2(1 - \gamma)^{i+1} - (1 - \delta + \alpha\delta)^{i+1}.$$

From this point on, we impose the following assumption,

$$\delta \geq \gamma > 0. \tag{91}$$

for future reference.

On the other hand, while both previous studies and causal observations support (91), we do not know in general whether $(\delta - \gamma - \delta\alpha) > 0$ or not. Notice that

$$\begin{aligned} (\delta - \gamma - \delta\alpha) &> 0 \\ \Leftrightarrow \delta(1 - \alpha) &> \gamma \quad . \\ \Leftrightarrow (1 - \alpha) &> \gamma/\delta \end{aligned} \tag{92}$$

The left hand side of the last expression is the labor share, or unity minus the capital share. The right hand side of the same expression is the ratio of the depreciation rate of residential housing relative to the depreciation rate of business capital. Thus, equation (92) compares the relative magnitude of two forces. One is the share of factors (which is labor in this model) other than business capital. The second force is the relative speed of business capital depreciation relative to the counterpart of another stock (which is residential housing in this model). Theoretically, there is no clear reason of why one should dominate the other. Since we cannot be sure whether $(\delta - \gamma - \delta\alpha) > 0$ or not, we cannot be sure whether $\left(\frac{\delta - \gamma}{\delta - \gamma - \delta\alpha}\right)$ is larger than unity or not.

In terms of the dynamics of $\delta_h(i)$, we can actually say a few more words. Notice that

$$\begin{aligned} &(x + z) \delta_h(i) \\ &= (x + z) (\varphi x^{i+1} + (1 - \varphi) z^{i+1}) \\ &= \varphi x^{i+2} + (1 - \varphi) z^{i+1} x + \varphi x^{i+1} z + (1 - \varphi) z^{i+2} \\ &= \delta_h(i + 1) + xz ((1 - \varphi) z^i + \varphi x^i) \\ &= \delta_h(i + 1) + xz \delta_h(i - 1). \end{aligned}$$

We re-state the result in the following lemma.

Lemma 3 $\delta_h(i)$ evolves according to the following formula,

$$\delta_h(i + 1) = (x + z) \delta_h(i) - xz \cdot \delta_h(i - 1). \tag{93}$$

Notice that for empirically plausible parameterization, $(x + z) = (1 - \gamma) + (1 - \delta + \alpha\delta) > 1$, whereas $xz = (1 - \gamma) \cdot (1 - \delta + \alpha\delta)$, hence $0 < xz < 1$.