System Reduction and Finite-Order VAR Solution Methods for Linear Rational Expectations Models[*]

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Abstract

This paper considers the solution of a large class of linear rational expectations (LRE) models cast in state-space form and their solution's characterization via finite-order VARs. Based on the method of undetermined coefficients, I propose a unified approach that uses a companion Sylvester equation to check the existence and uniqueness of a solution to the canonical LRE model in finite-order VAR form and to simplify its characterization. Solving LRE models by this procedure is straightforward to implement, general in its applicability, efficient in the use of computational resources, and can be handled easily with standard matrix algebra. I also explore how to correctly recover theory-consistent monetary policy shocks from observed data. An application to the workhorse New Keynesian model with accompanying Matlab codes is provided to illustrate the practical implementation of the methodology. I argue that empirical evidence on the monetary transmission mechanism and on monetary policy shocks from incorrectly-specified structural VARs (in terms of lags, zero identification restrictions, etc.) should be interpreted carefully as it may not have a proper structural interpretation.

JEL codes: C32, C62, C63, E37

1 Introduction

The solution of linear or linearized rational expectations (LRE) models is an important part of modern macroeconomics. They are widely used to study the propagation mechanism of economic shocks, for identification, and to provide economic evaluation of policy changes. Many rational expectations macro models can be cast as a linear system of expectational difference equations. The linearity of the system may be a feature of the model itself but often is simply achieved from the first-order approximation of a Dynamic Stochastic General Equilibrium (DSGE) model.

Blanchard and Kahn (1980) established the conditions for existence and uniqueness of a solution to the LRE model (see, among others, the related contributions of Broze *et al.* (1985, 1990), King and Watson (1998), Uhlig (1999), and Klein (2000)). Maximum likelihood and Bayesian estimation methods can be used on LRE models with a unique solution to achieve a constrained fit of the data. However, the theory may fail in fitting the data satisfactorily because of misspecification—the cross-equation restrictions imposed may be at odds with the *true* data-generating process (DGP). Even under the null hypothesis that the LRE model is correctly specified, it may still be the case that the theory suffers from weak identification problems (Martínez-García *et al.* (2012), Martínez-García and Wynne (2014)).

Structural VAR models in the spirit of Sims (1980) provide a framework with which to investigate and organize the evidence from the observable variables that, in principle, is largely devoid of the restrictions implied by theory (the LRE model) and more flexible to fit the data. The work, among others, of Fernández-Villaverde *et al.* (2007), Ravenna (2007), and more recently Franchi and Paruolo (2015) has shed new light on the mapping between the unique LRE model solution and a corresponding VAR form. To be more precise, this strand of the literature explores conditions under which the unique solution of the LRE model—when it exists—can be properly represented in VAR form, as this facilitates the recovery of structural shocks as well as empirical inference and validation of LRE models that are brought to the data.

The structural shocks of the LRE model cannot always be recovered even when a VAR representation of the unique LRE solution can be obtained due to lack of fundamentalness. Non-fundamentalness means that the observed variables do not contain enough information to recover the unobserved structural shocks (Hansen and Sargent (1980)). An LRE model solution is said to be fundamental if the structural moving average (MA) representation of the observed variables can be inverted. If the LRE model solution is fundamental (assuming the LRE model itself is also correctly specified), then the observed variables have a VAR representation in the structural shocks—implying that the structural shocks can be recovered by estimating a VAR with the observed variables and that their corresponding impulse response functions can be correctly inferred.[1]

When the number of structural shocks is equal to the number of observables, the fundamentalness property of the unique solution to the LRE model can be checked with the 'poor man's invertibility condition' of Fernández-Villaverde *et al.* (2007). Ravenna (2007) proposes a 'unimodularity condition' to determine when the VAR representation of the unique LRE solution is of finite order—since an inverted structural MA representation of the unique LRE model solution can also take a VAR(∞) form and this introduces a truncation error when cast as a finite-order structural VAR (Inoue and Kilian (2002)).

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[1] Even if the unique model solution has a VAR representation that is fundamental—in that it permits the exact recovery of the structural shocks and the characterization of their corresponding impulse response functions—it is worth noting that this does not ensure that the matrices of composite coefficients that describe the dynamics for the VAR form will uniquely identify all the structural parameters of a well-specified LRE model. Therefore, policy evaluation and even model comparison are still subject to the perils of identification failure noted by Martínez-García *et al.* (2012) and Martínez-García and Wynne (2014). In any event, the treatment of weak/partial identification and even misspecification falls outside the scope of this paper.

Franchi and Paruolo (2015) show that if the state-space representation of the LRE solution is minimal, then both the 'poor man's invertibility condition' of Fernández-Villaverde *et al.* (2007) and the 'unimodularity condition' of Ravenna (2007) are necessary and sufficient to ensure fundamentalness and the existence of an exact finite-order VAR form for the solution of the LRE model. In their paper, Franchi and Paruolo (2015) argue that when the state-space representation is non-minimal, the 'poor man's invertibility condition' and the 'unimodularity condition' are not necessary.[2]

In this paper, I complement the existing literature by proposing an alternative approach for determining existence and uniqueness and characterizing the unique solution in finite-order VAR form for a large class of LRE models via a pair of companion matrix equations—a quadratic matrix equation and a Sylvester matrix equation. First, LRE models that include backward-looking and forward-looking features with one or more lags and leads can be reduced to the canonical form of an expectational first-order difference equation without backward-looking terms. System reduction from the general form of the LRE model to the canonical form is achieved by solving a companion quadratic matrix equation—if at least one solution exists.

Second, the well-known method of undetermined coefficients can then be used to solve the canonical forward-looking part of the LRE model. Conditions under which a finite-order VAR representation of the canonical LRE model solution can be obtained and a simple (yet efficient) algorithm to compute it can be derived from a companion Sylvester matrix equation. The final step simply requires reversing the transformation of the system utilized in the first step in order to recover the solution representation for the general form of the LRE model.

The initial step of system reduction involves the solution of a quadratic matrix equation, and permits generalizing the approach (and the implementation) proposed in this paper to cover a wide range of LRE models (Binder and Pesaran (1995, 1997)). However, the key contribution of the paper is that the characterization of the finite-order VAR solution of the canonical LRE model arises naturally from the solution of a Sylvester matrix equation. I propose a simple approach based on this companion Sylvester equation to check for and identify unique LRE solutions in finite-order VAR form and a simple algorithm to compute such solutions. The conditions that verify existence and uniqueness of a finite-order VAR for the canonical LRE model solution also ensure its fundamentalness.

These tools are meant to be used by macroeconomists who deal with LRE models in their theoretical or applied work and who need to determine if the observable (endogenous) variables admit an exact finite-order VAR representation that is also fundamental. I illustrate the practical use of this novel approach with the workhorse three-equation New Keynesian model—showing how the procedure can be used to derive the finite-order VAR representation of the unique LRE model solution, to establish its existence and uniqueness, and to make economically-relevant inferences about the New Keynesian transmission mechanism to recover structural shocks and explore their propagation patterns. Moreover, I also contribute to the ongoing debate on the assumptions needed to correctly recover theoretically-consistent monetary policy shocks through structural VARs (Carlstrom *et al.* (2009)).

The rest of the paper proceeds as follows: Section 2 describes the system reduction method to decouple

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[2] The state-space form is called minimal if the dimension of the vector of forcing variables is as small as possible (Kailath (1980)). Fundamentalness and the existence of a finite-order VAR representation of the LRE model solution that is non-minimal can still be asserted by first transforming the state-space form to a minimal representation and then applying the existing conditions. Franchi and Paruolo (2015) explore necessary and sufficient conditions that are valid when the state-space form is non-minimal based on the possibility of exploiting cancellations (as in related problems from systems theory), bypassing the step of transforming the LRE model to its minimal state-space form first.

the backward-looking and forward-looking parts of the general form LRE model and shows how to use the method of undetermined coefficients to characterize the linear state-space form solution of the (canonical) forward-looking part of the LRE model. Section 3 describes the mapping of the LRE model solution into finite-order VAR form via a companion Sylvester equation. This section also discusses the conditions under which a unique finite-order VAR solution can be attained from the companion Sylvester equation as well as the algorithms available to compute it. Section 4 applies the method to a policy-relevant illustration on the effects of monetary policy on inflation determination based on the workhorse three-equation New Keynesian model with which I also illustrate the computational efficacy of the approach. Furthermore, I discuss the fundamentalness of the finite-order VAR solution and what it means for the ongoing debate on the assumptions required to recover structural monetary shocks from VARs. Section 5 then concludes.

The Appendix provides additional technical details on the system reduction approach used in this paper to isolate the forward-looking part of the LRE model in general form—including a generalized eigenvalue problem algorithm to implement it. The Appendix discusses in detail the derivation of the MA representation of the LRE model solution when cast in linear state-space form. It also discusses other technical aspects related to the state-space form of the solution and describes the large class of LRE models for which the procedure can be utilized. The procedure itself is easily cast in an algorithmic form, and a collection of Matlab implementation codes is provided with the paper.[3]

2 The LRE Model

Going from the structural relationships implied by the LRE model to a reduced-form solution requires explicit assumptions about the formation of expectations and the stochastic process attached to the exogenous forcing variables. The structural relationships that characterize the LRE model are always *true* according to theory, but the reduced-form solution will depend on those other assumptions as well. Here, I consider LRE models where expectations are fully rational and the exogenous forcing variables are assumed to follow a VAR process. Under rational expectations, agents understand the structure of the economy and formulate expectations optimally incorporating all available information.[4]

Under all these assumptions, a mapping can arise between the reduced-form solution of the LRE model and a structural VAR representation for the (observable) endogenous variables that explains why VARs appear to fit the data well. The VAR representation of the reduced-form solution also provides researchers with more bite to investigate the propagation of shocks than an unrestricted VAR does. In this paper, I explore the connection between the reduced-form solution of LRE models and structural VARs to bridge the gap between theoretical and applied work towards a more unified approach.

A large class of LRE models can be cast into a first-order expectational difference system of equations, featuring forward- and backward-looking dynamics. The first-order expectational difference equations cap-

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[3] All codes for this paper are available in my website: https://sites.google.com/site/emg07uw/ and can be downloaded directly using the following link: https://sites.google.com/site/emg07uw/econfiles/LRE_model_solution.zip?attredirects=0.

   Straightforward manipulations of those codes can be made to adapt them to other LRE models that can be cast in the first-order form investigated in this paper—users of the codes are asked to include a citation of this paper in their work. The Matlab programs and functions appear free of errors, however I do appreciate all feedback, suggestions or corrections that you may have. While users are free to copy, modify and use the code for their work, I do not assume any responsibility for any remaining errors or for how the codes may be used or misused by users other than myself.

[4] The idea of rational expectations can be traced to the seminal work by Muth (1961). Lucas (1976) and Sargent (1980) were among the leading economists that rejected *ad hoc* assumptions on the formation of expectations and advocated the adoption of rational expectations that became prevalent in modern macroeconomics since the 80s.

ture the structural relationships between a subset of k endogenous variables $W_t = (w_{1t}, w_{2t}, ..., w_{kt})^T$ and k exogenous forcing variables $X_t = (x_{1t}, x_{2t}, ..., x_{kt})^T$ as follows:

$$W_t = \Phi_1 W_{t-1} + \Phi_2 \mathbb{E}_t [W_{t+1}] + \Phi_3 X_t, \tag{1}$$

$$X_t = A X_{t-1} + B \epsilon_t, \tag{2}$$

where Φ_1, Φ_2, and Φ_3 are conforming $k \times k$ square matrices. The structural relationships of the first-order LRE model given by (1) are completed with the standard VAR(1) specification for the vector of k forcing variables X_t in (2), where A is a $k \times k$ matrix that has all its eigenvalues inside the unit circle ensuring the stationarity of the stochastic process for the forcing variables, B is a $k \times k$ positive definite matrix of variances and covariances, and ϵ_t is the corresponding column-vector of dimension k of independent and identically distributed (i.i.d.) zero-mean innovations.

The LRE model involves $k + p$ endogenous variables $\left(W_t^T, \widetilde{W}_t^T \right)^T$ where the remaining $p \geq 0$ endogenous variables given by $\widetilde{W}_t = (\widetilde{w}_{1t}, \widetilde{w}_{2t}, ..., \widetilde{w}_{pt})^T$ can simply be expressed as functions of W_t and X_t, possibly including lags and expectations of their leads too.[5] Therefore, the solution for the subset of endogenous variables W_t derived from the system in $(1) - (2)$ suffices to map out the relationship between the remaining endogenous variables $\widetilde{W}_t$ and the forcing variables X_t.[6]

The system in $(1) - (2)$ can be further generalized to capture LRE models including more than one lead and lag of the endogenous and forcing variables in the specification, as explained in the Appendix. Hence, the compact form of the LRE model given by $(1) - (2)$ can be generalized to investigate a large class of LRE models.

2.1 Decoupling Backward-Looking and Forward-Looking Terms

Building on Broze *et al.* (1985, 1990) and Binder and Pesaran (1995, 1997), I adopt a simple transformation of the compact form of the LRE model in $(1) - (2)$ that achieves a system reduction excluding all backward-looking terms from the expectational difference system and then I work out the full LRE model solution by parts. For a given $k \times k$ matrix Θ, the transformation of the subset of endogenous variables W_t given by $W_t \equiv Z_t + \Theta W_{t-1}$ implies that the expectational difference system in (1) can be rewritten as:

$$\begin{aligned} Z_t + \Theta W_{t-1} &= \Phi_1 W_{t-1} + \Phi_2 \mathbb{E}_t [Z_{t+1} + \Theta W_t] + \Phi_3 X_t \\ &= \Phi_1 W_{t-1} + \Phi_2 [\mathbb{E}_t (Z_{t+1}) + \Theta (Z_t + \Theta W_{t-1})] + \Phi_3 X_t, \end{aligned} \tag{3}$$

which becomes,

$$(I_k - \Phi_2 \Theta) Z_t = \Phi_2 \mathbb{E}_t (Z_{t+1}) + \left(\Phi_2 \Theta^2 - \Theta + \Phi_1 \right) W_{t-1} + \Phi_3 X_t. \tag{4}$$

From here, this condition follows:

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[5] Here, the number of endogenous variables is at least the same or higher than the number of exogenous forcing variables. Hence, if the solution to $(1) - (2)$ can be represented in finite-order VAR form, it can be exploited to exactly recover every one of the structural shocks forcing the LRE model. I do not consider explicitly the case of partial recovery that arises when the number of endogenous variables is smaller than the number of exogenous forcing variables, though. In that case, in general, only linear combinations of the structural shocks can be recovered but not the shocks themselves (or even a subset of them).

[6] The selection of the subset of k endogenous variables W_t can be significant, for instance, for the identification of estimated structural parameters, as noted in Martínez-García *et al.* (2012) and Martínez-García and Wynne (2014)). These questions, however, go beyond the scope of the current paper.

Condition 1 *A system reduction that excludes the backward-looking terms in (1) can be attained by choosing a $k \times k$ matrix Θ to satisfy that*

$$P(\Theta) = \Phi_2 \Theta^2 - \Theta + \Phi_1 = 0_k, \tag{5}$$

where 0_k is a $k \times k$ matrix of zeroes.

This transformation uncouples the solution of W_t implied by (1) into a backward-looking part, $W_{tb} \equiv \Theta W_{t-1}$, and a forward-looking part, $W_{tf} \equiv Z_t$, such that $W_t \equiv W_{tb} + W_{tf}$. Hence, to solve the compact form of the LRE model, one first needs to determine a matrix Θ solving the quadratic matrix equation in (5) to characterize the backward-looking part of the solution and then reduce the expectational difference system in (1) to its canonical (purely forward-looking) expectational difference part.

Binder and Pesaran (1995, 1997) establish the necessary and sufficient conditions under which real-valued solutions for Θ satisfying the quadratic matrix equation in (5) exist. They also provide a straightforward iterative algorithm to compute its stable solution. A discussion of an alternative algorithm to characterize the stable solution Θ based on the generalized eigenvalue problem can be found in the Appendix.

After decoupling, the vector of k transformed endogenous variables, $Z_t \equiv W_t - \Theta W_{t-1}$, follows a canonical first-order forward-looking expectational difference system of the following form:

$$\Gamma_0 Z_t = \Gamma_1 \mathbb{E}_t [Z_{t+1}] + \Gamma_2 X_t, \tag{6}$$

where $\Gamma_0 \equiv (I_k - \Phi_2 \Theta)$, $\Gamma_1 \equiv \Phi_2$, and $\Gamma_2 \equiv \Phi_3$ are conforming $k \times k$ matrices. Whenever Γ_0 is nonsingular, the canonical system of structural relationships implied by (6) can be rewritten as:

$$Z_t = F \mathbb{E}_t [Z_{t+1}] + G X_t, \tag{7}$$

where $F \equiv (\Gamma_0)^{-1} \Gamma_1$ and $G \equiv (\Gamma_0)^{-1} \Gamma_2$.

The invertibility of Γ_0 required to go from (6) to (7) depends on the choice of the matrix Θ. Proposition 2 in Binder and Pesaran (1997) discusses conditions under which the solution can be characterized analytically and then, more specifically, provides sufficient conditions under which $(I_k - \Phi_2 \Theta)$ would be non-singular. Whenever the Binder and Pesaran (1997) conditions are satisfied, the matrix Γ_0 can be shown to be nonsingular and invertible. The Binder and Pesaran (1997) conditions are only sufficient (not necessary), but I find that most well-specified economic LRE models produce a matrix Γ_0 that is indeed nonsingular.

Hence, in the remainder of this paper, I take the specification in (7) to constitute the relevant benchmark to describe the canonical forward-looking part of the LRE model solution.

2.2 The 'Method of Undetermined Coefficients' Solution

The conventional approach to characterize a solution for the LRE model is laid out in Blanchard and Kahn (1980), which also provide conditions to check the existence and uniqueness of the solution. Blanchard and Kahn's (1980) method was further refined and extended by Broze *et al.* (1985, 1990), King and Watson (1998), Uhlig (1999), and Klein (2000), among others, to obtain the solution in more general settings. Other popular solution methods applied to LRE models include the method of rational expectational errors of Sims (2002) (see also Lubik and Schorfheide (2003)) and the method of undetermined coefficients of Christiano (2002).

Assuming that a solution exists and is unique, then the solution to the canonical forward-looking part of the LRE model in (7) can be written in linear state-space form as follows:

$$X_t = AX_{t-1} + B\epsilon_t, \tag{8}$$

$$Z_t = CX_{t-1} + D\epsilon_t, \tag{9}$$

where A, B, C, and D are conforming $k \times k$ matrices. Equation (8) simply re-states (2) on the dynamics of the exogenous forcing variables X_t, while (9) indicates that current innovations and lagged exogenous forcing variables are mapped into the transformed endogenous variables Z_t in the solution of the LRE model.

In order to pin down the solution, I need to relate the unknown matrices of composite coefficients C and D to the known composite matrices that describe the structural relationships of the LRE model (F, G) in (7) and those that describe the stochastic process of the forcing variables (A, B) re-stated in (8). Using the method of undetermined coefficients, such a solution can be characterized via the solution of a companion Sylvester matrix equation and, under some conditions, it can be shown to have an exact finite-order VAR representation.

Step 1. Using (9) shifted one period ahead to replace Z_{t+1} in the purely forward-looking system given in (7) implies that:

$$Z_t = [FC + G] X_t \tag{10}$$

$$= [FC + G] AX_{t-1} + [FC + G] B\epsilon_t, \tag{11}$$

where the second equality arises from replacing X_t out using (8). The forward-looking LRE model solution conjectured in (9) can be matched with (11) to link the unknown solution matrices C and D to the known matrices—composites of structural parameters—that describe the structural relationships of the LRE model (F, G, A, B).

By the method of undetermined coefficients, it follows that the conforming square matrices C and D that characterize (9) in the solution must satisfy the following pair of conditions:

$$C = [FC + G] A, \tag{12}$$

$$D = CA^{-1}B. \tag{13}$$

The eigenvalues of matrix A must be inside the unit circle by construction for the VAR(1) process associated with the forcing variables to be stationary. I also assume that zero is not an eigenvalue of matrix A as that ensures the inverse matrix A^{-1} in (13) exists and is well-defined according to the invertible matrix theorem (Strang (2016)).

As a result, the existence and uniqueness of a solution to C that satisfies (12) also pins down D through (13) ensuring the existence and uniqueness of the full solution to the forward-looking part of the LRE model given by the linear state-space form in (8) − (9). Hence, assuming Γ_0 is invertible as suggested before, I find that solving the companion Sylvester matrix equation given by (12) to obtain C is enough to characterize the solution to the forward-looking part of the LRE model in (8) − (9).

Step 2. Using (8) and the invertibility of A, I can write X_{t-1} as $X_{t-1} = A^{-1}(X_t - B\epsilon_t)$. Replacing

this expression in (9), I infer that:

$$Z_t = CA^{-1}X_t + \left[D - CA^{-1}B\right]\epsilon_t. \tag{14}$$

Then, it follows from condition (13) that characterizes the matrix D using the method of undetermined coefficients that the term related to the vector of innovations ϵ_t must drop out from (14). As a result, the solution of the forward-looking part of the LRE model implies a straightforward linear mapping from the vector of exogenous forcing variables X_t to the vector of transformed endogenous variables Z_t where,

$$Z_t = CA^{-1}X_t, \tag{15}$$

if a matrix C exists and condition (13) holds.

Whenever a unique C exists which is also shown to be invertible, equation (15) implies that $X_t = AC^{-1}Z_t$ given that A is already invertible by construction. Shifting this expression one period back and replacing it in (9), I obtain the following VAR(1) specification to characterize the solution of the forward-looking part of the LRE model:

$$Z_t = CAC^{-1}Z_{t-1} + CA^{-1}B\epsilon_t, \tag{16}$$

where I replaced D using condition (13).

The existence and uniqueness of the matrix D follows naturally under condition (13) from the existence and uniqueness of a matrix C that solves the Sylvester matrix equation given by condition (12). However, I also find that the solution of the forward-looking part of the LRE model has the finite-order VAR representation given by (16) whenever a unique matrix C solving the Sylvester matrix equation exists which can also be shown to be invertible. Therefore, the characterization of the finite-order VAR solution in (16) depends on the invertibility of Γ_0 as indicated before, but also on the existence, uniqueness, and invertibility of the matrix solution C arising from condition (12).

Rewriting condition (12), the characterization, existence, and uniqueness of a finite-order VAR solution can be summarized as follows:

Lemma 1 *If Γ_0 is invertible, a VAR(1) representation of the solution to the first-order (purely forward-looking) expectational difference system of equations in (7) can be obtained by solving a companion Sylvester matrix equation in C:*

$$FCA - C = H, \tag{17}$$

where

$$F \equiv \left(\Gamma_0\right)^{-1}\Gamma_1, \ H \equiv -GA = -\left(\Gamma_0\right)^{-1}\Gamma_2 A. \tag{18}$$

If a unique matrix C exists and is invertible, the VAR(1) representation of the solution is given by (16).

The proof of this lemma follows directly from the derivation of conditions $(12) - (13)$ by the method of undetermined coefficients, as discussed above.

Step 3. Then, the solution of the compact (first-order) form of the LRE model can be obtained by combining its backward- and forward-looking parts—i.e., $W_t \equiv W_{tb} + W_{tf}$ where the backward-looking part, $W_{tb} \equiv \Theta W_{t-1}$, follows from the solution to the quadratic matrix equation in (5) and the forward-looking part, $W_{tf} \equiv Z_t \equiv W_t - \Theta W_{t-1}$, is defined by the state-space solution in $(8) - (9)$ given conditions $(12) - (13)$.

Under the assumptions that a solution Θ for the companion quadratic matrix equation exists and that a unique solution C for the companion Sylvester matrix equation exists and is invertible (as stated in Lemma 1), the implication is that the solution of the compact form of the LRE model is given by a VAR(2) process of the following form:

Corollary 1 *If a unique matrix C exists that solves the companion Sylvester matrix equation of Lemma 1 and this matrix C is also invertible, the VAR(2) representation of the first-order LRE model solution for the vector of endogenous variables W_t is given by:*

$$W_t = \Psi_1 W_{t-1} + \Psi_2 W_{t-2} + \Psi_3 \epsilon_t, \tag{19}$$

where the corresponding coefficient matrices are $\Psi_1 \equiv \left(\Theta + CAC^{-1}\right)$, $\Psi_2 \equiv -CAC^{-1}\Theta$, and $\Psi_3 \equiv CA^{-1}B$.

The derivation of this corollary follows naturally from the definition of the transformation of the endogenous variables as $Z_t = W_t - \Theta W_{t-1}$ and the derivation of a VAR(1) representation for Z_t implied under the terms of Lemma 1. The matrix Ψ_3 is not necessarily going to be positive semi-definite and symmetric, so it does not have the standard interpretation of a variance-covariance matrix unlike the matrix B.[7] In turn, Corollary 1 implies that Ψ_3 is a linear transformation of the variance-covariance matrix of the stochastic process, B, where the mapping is determined by CA^{-1} (which depends on the solution C of the companion Sylvester matrix equation).

Furthermore, the structural shock innovations of the LRE model can be exactly recovered—making the finite-order VAR form in (19) consistent with fundamentalness (Hansen and Sargent (1980)). Assuming a unique matrix C exists and is invertible, the canonical LRE model solution in (16) can be re-written in terms of the innovations as $\epsilon_t = B^{-1}A\left(C^{-1}Z_t - AC^{-1}Z_{t-1}\right)$, given that the variance-covariance matrix B is invertible by construction.[8] Hence, undoing the transformation of the endogenous variables, I obtain the following expression in terms of the observable endogenous variables W_t:

$$\epsilon_t = B^{-1}A\left[C^{-1}W_t - \left(C^{-1}\Theta + AC^{-1}\right)W_{t-1} + AC^{-1}\Theta W_{t-2}\right]. \tag{20}$$

Hence, the fundamentalness of the unique finite-order VAR solution can be summarized as follows:

Corollary 2 *If a unique matrix C exists that solves the companion Sylvester matrix equation of Lemma 1 and this matrix C is also invertible, the VAR(2) representation of the first-order LRE model solution for the vector of endogenous variables W_t is fundamental and the realization of the shocks can be recovered exactly using equation (20).*

The proof of this corollary follows directly from simple algebraic manipulations of the finite-order VAR(2) representation of the solution, as described above.

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[7] To test whether Ψ_3 has the properties of a variance-covariance matrix, I can use the *chol* function in Matlab. If *chol* returns a second argument that is zero from $[R, p] = chol\left(\Psi_3\right)$, then the matrix is confirmed to be symmetric and—in this case—also positive definite.

[8] A positive definite variance-covariance matrix is invertible. However, a variance-covariance matrix that is positive semidefinite but not positive definite would not be invertible.

3 The Finite-Order VAR Representation

If a solution C exists for the companion Sylvester matrix equation given by (17), then a solution exists for the canonical forward-looking part of the LRE model given in state-space form by (8) − (9). According to Lemma 1, such a solution can be represented with a finite-order VAR whenever C is unique and also shown to be invertible. Hence, the main methodological contribution of this paper is that is shows how to solve a large class of LRE models that can be cast into the compact first-order form given by (1) − (2) via a companion Sylvester matrix equation. The corollary of the approach I propose is that under some conditions applied to the solution of the companion Sylvester matrix equation, one can check whether the LRE model solution admits a finite-order VAR representation. In this section I discuss the characterization of the solution to the Sylvester matrix equation in (17) and provide an overview of efficient algorithms to compute it numerically.

3.1 Characterization of the Solution

Equation (17) proposes a companion Sylvester matrix equation—i.e., $FCA - C = H$ with $F, A, H \in \mathbb{R}^{k \times k}$ given and $C \in \mathbb{R}^{k \times k}$ to be determined—as an alternative to characterize the solution of the forward-looking part of the LRE model. The Sylvester matrix equation is well-known in stability and control theory and its applications.[9] Using the Kronecker (tensor) product notation and the properties of the vectorization operator, vec, I can re-write Sylvester's matrix equation in its standard form as a linear system of equations:

$$\mathcal{A}vec\left(C\right) = vec\left(H\right), \tag{21}$$

$$\mathcal{A} := \left[\left(A^T \otimes F\right) - I_{k^2}\right], \tag{22}$$

where $\otimes$ denotes the Kronecker product.[10] In this way, the Sylvester matrix equation is represented by a linear system of dimension $k^2 \times k^2$ conformed by k^2 equations in k^2 unknown variables (where the unknowns correspond to the elements of the matrix C).

Having transformed the Sylvester equation into the linear system given by (21) − (22), well-known matrix algebra results suffice to determine the following criteria for the existence and uniqueness of a solution C to

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[9] The Sylvester matrix equation is extensively discussed in Lancaster and Tismenetsky (1985, Chapter 12). Additional useful references on the characterization of the solution to the Sylvester matrix equation include Horn and Johnson (1991) and Jiang and Wei (2003).

[10] For a given matrix $X \in \mathbb{R}^{k \times k}$, express $X = [X_{*1} \ X_{*2} \ ... \ X_{*k}]$ where $X_{*j} \in \mathbb{R}^k$, $j = 1, 2, ..., k$. Then, the vectorization associated to matrix X defines the following vector-valued function:

$$\begin{bmatrix} X_{*1} \\ X_{*2} \\ ... \\ X_{*k} \end{bmatrix} \in \mathbb{R}^{k^2},$$

which is denoted $vec\left(X\right)$. The vectorization operation is linear, i.e. $vec\left(\alpha X + \beta Y\right) = \alpha vec\left(X\right) + \beta vec\left(Y\right)$ for any $X, Y \in \mathbb{R}^{k \times k}$ and $\alpha, \beta \in \mathbb{R}^k$. If $X = [x_{ij}]_{i,j=1}^{k} \in \mathbb{R}^{k \times k}$ and $Y \in \mathbb{R}^{k \times k}$, then the Kronecker (tensor) product of X and Y, written $X \otimes Y$, is defined to be the partitioned matrix:

$$X \otimes Y = [x_{ij}Y] = \begin{bmatrix} x_{11}Y & x_{12}Y & ... & x_{1k}Y \\ x_{21}Y & x_{22}Y & ... & x_{2k}Y \\ \vdots & \vdots & & \vdots \\ x_{k1}Y & x_{k2}Y & & x_{kk}Y \end{bmatrix} \in \mathbb{R}^{k^2 \times k^2}.$$

Proposition 4 in Chapter 12.2 of Lancaster and Tismenetsky (1985) shows that the vectorization operation is closely related to the Kronecker product as follows: If $X, Y, Z \in \mathbb{R}^{k \times k}$, then $vec\left(XYZ\right) = \left(Z^T \otimes X\right) vec\left(Y\right)$.

the companion Sylvester matrix equation:

Proposition 1 *Let $F, A, H \in \mathbb{R}^{k \times k}$. Then, it follows that:*

(a) (Existence) The Sylvester equation in (17) has at least one solution $C \in \mathbb{R}^{k \times k}$ if and only if $rank\,[\mathcal{A}\;vec\,(H)] = rank\,[\mathcal{A}]$.

(b) (Uniqueness) The Sylvester equation in (17) has a unique solution $C \in \mathbb{R}^{k \times k}$ if and only if $rank\,[\mathcal{A}] = k^2$. That is, the solution is unique if and only if $\mathcal{A}$ has full rank. Then, $\mathcal{A}$ is nonsingular and invertible implying that the unique solution to the Sylvester matrix equation can be recovered as $vec\,(C) = \mathcal{A}^{-1}vec\,(H)$.

Proof. (a) Trivially it follows that $rank\,[\mathcal{A}\;vec\,(H)] \geq rank\,[\mathcal{A}]$. If there is a solution $vec\,(C) = [\bar{c}_1\;\bar{c}_2\;...\;\bar{c}_{k^2}]^T$ for the linear system given by $(21) - (22)$, then $\sum_{i=1}^{k^2} \mathcal{A}_{*i}\bar{c}_i = vec\,(H)$ where $\mathcal{A}_{*1}, \mathcal{A}_{*2}, ..., \mathcal{A}_{*k^2}$ denote the corresponding columns of the matrix $\mathcal{A}$. Hence, $vec\,(H)$ is a linear combination of the columns of $\mathcal{A}$ and, as a result, the rank of the augmented matrix $[\mathcal{A}\;vec\,(H)]$ cannot be different than the rank of $\mathcal{A}$—because for $rank\,[\mathcal{A}\;vec\,(H)] > rank\,[\mathcal{A}]$ to be true, $vec\,(H)$ needs to be linearly independent from the columns of $\mathcal{A}$ and that contradicts $vec\,(C)$ being a solution.

(b) If the square matrix $\mathcal{A}$ has full rank—and, therefore, is nonsigular and invertible—the linear system in $(21) - (22)$ has a unique solution given by $vec\,(C) = \mathcal{A}^{-1}vec\,(H)$. The converse statement follows naturally as well. If the linear system has a unique solution, then $\mathcal{A}$ must have full rank and be nonsingular. Otherwise, at least one column in $\mathcal{A}$ is not linearly independent from the rest of the columns and can be written as a linear combination of them. Hence, for any given solution $vec\,(C)$ defined over the linearly independent columns, another different solution exists including non-trivially the linearly dependent columns of $\mathcal{A}$ as well. The existence of more than one solution then contradicts the uniqueness assumption. ∎

Proposition 1 characterizes the solution C to the Sylvester equation in (17) and, by extension, determines conditions for the existence and uniqueness of the solution to the forward-looking part of the LRE model. The two rank conditions stated in this proposition depend solely on the properties of the matrices $F, A, H \in \mathbb{R}^{k \times k}$ that describe the structural relationships of the first-order LRE model. The uniqueness rank condition implies a solution of the form $vec\,(C) = \mathcal{A}^{-1}vec\,(H)$ for the Sylvester matrix equation and, naturally, that proves existence as well. If the uniqueness rank condition is violated, the existence rank condition determines whether there is no solution to the companion Sylvester matrix equation—if $rank\,[\mathcal{A}\;vec\,(H)] \neq rank\,[\mathcal{A}]$— or whether multiple solutions exist—if $rank\,[\mathcal{A}\;vec\,(H)] = rank\,[\mathcal{A}] < k^2$. In the latter case, it can be shown that the number of linearly independent solutions is determined by the dimension of the kernel of $\mathcal{A}$.

The characterization of the linearly independent solutions of the companion Sylvester matrix equation whenever a solution exists but is not unique can be found in Theorem 12.5.1, Theorem 12.5.2 and Corollary 12.5.1 of Lancaster and Tismenetsky (1985). Focusing on the case of interest for this paper where a solution to the Sylvester matrix equation in (17) exists and is unique, the full rank condition on $\mathcal{A}$ can be expressed in terms of the eigenvalues of F and A as follows:

Proposition 2 *Let $F, A \in \mathbb{R}^{k \times k}$ be given. Let $\lambda_1, ..., \lambda_k$ be the eigenvalues of F and $\mu_1, ..., \mu_k$ the eigenvalues of A. Then, for any matrix $H \in \mathbb{R}^{k \times k}$, it follows that the companion Sylvester matrix equation in (17) has a unique solution if and only if $\lambda_i \mu_j \neq 1$ for all $i, j = 1, ..., k$. In other words, the Sylvester matrix equation has a unique solution $C \in \mathbb{R}^{k \times k}$ if and only if the matrices F and A^{-1} have no eigenvalues in common.*

Proof. The eigenvalues of A are the same as those of its transpose A^T. Given that and the properties of the Kronecker product, the eigenvalues of $\left(A^T \otimes F\right)$ are the k^2 numbers defined by the product between

the eigenvalues of F and A, i.e., $\lambda_i \mu_j$ for all $i, j = 1, ..., k$. Then, the eigenvalues of $\mathcal{A} := \left[\left(A^T \otimes F \right) - I_{k^2} \right]$ are simply the k^2 numbers $\lambda_i \mu_j - 1$ for all $i, j = 1, ..., k$. By Proposition 1, the existence and uniqueness of a solution to the Sylvester matrix equation requires $\mathcal{A}$ to be nonsingular (and have full rank). The matrix $\mathcal{A}$ is nonsingular if and only if all its eigenvalues are nonzero, i.e. if and only if $\lambda_i \mu_j - 1 \neq 0$ for all $i, j = 1, ..., k$. Re-arranging the nonzero conditions on the eigenvalues, it follows that $\lambda_i \neq \frac{1}{\mu_j}$ for all $i, j = 1, ..., k$. Given that the eigenvalues of A^{-1} are $\frac{1}{\mu_1}, ..., \frac{1}{\mu_k}$ while those of F are $\lambda_1, ..., \lambda_k$, a unique solution is said to exist if and only if the matrices F and A^{-1} have no eigenvalues in common. ∎

According to Proposition 2, the companion Sylvester matrix equation for the canonical forward-looking part of the LRE model has a unique solution C for each matrix H if and only if F and A^{-1} have no eigenvalues in common. The Sylvester matrix operator $S : \mathbb{R}^{k \times k} \to \mathbb{R}^{k \times k}$ can be defined as follows:

$$S(C) = FCA - C, \tag{23}$$

where $F, A \in \mathbb{R}^{k \times k}$ are given and $C \in \mathbb{R}^{k \times k}$ is the solution to be identified. Then, the Sylvester matrix equation can simply be written as $S(C) = H$ for any given matrix $H \in \mathbb{R}^{k \times k}$.

The k^2 eigenvalues of the Sylvester operator $S(C)$ are $\lambda_i \mu_j - 1$, for all $i, j = 1, ..., k$, where $\lambda_1, ..., \lambda_k$ are the eigenvalues of F, and $\mu_1, ..., \mu_k$ are the eigenvalues of A. Let v_i be the right eigenvector of F associated with the eigenvalue λ_i such that $Fv_i = \lambda_i v_i$ for all $i = 1, ..., k$. Let w_j be the left eigenvector of A associated with the eigenvalue μ_j such that $w_j^T A = \mu_j w_j^T$ for all $j = 1, ..., k$. Then, for any $i, j = 1, ..., k$, $C = v_i w_j^T$ is an eigenvector matrix of the Sylvester matrix operator $S(C)$ associated with its eigenvalue $\lambda_i \mu_j - 1$. It follows from here that the Sylvester matrix operator can be expressed as:

$$\begin{aligned} S(C) = FCA - C &= F \left(v_i w_j^T \right) A - v_i w_j^T \\ &= (Fv_i) \left(w_j^T A \right) - v_i w_j^T \\ &= (\lambda_i v_i) \left(\mu_j w_j^T \right) - v_i w_j^T \\ &= (\lambda_i \mu_j - 1) v_i w_j^T \\ &= (\lambda_i \mu_j - 1) C. \end{aligned} \tag{24}$$

Hence, it can be shown that the Sylvester matrix operator $S(C)$ must be nonsingular whenever $\lambda_i \mu_j \neq 1$ for all $i, j = 1, ..., k$. In other words, $S(C)$ is nonsingular if and only if the solution to the Sylvester equation exists and is unique.

The eigenvalues of A, i.e. $\mu_1, ..., \mu_k$, are all inside the unit circle by construction to ensure the stochastic process for the forcing variables is stationary and none of those eigenvalues is 0 so A is invertible according to the invertible matrix theorem (Strang (2016)). However, the conditions for existence and uniqueness of the Sylvester matrix equation solution stated in Proposition 2 do not require the eigenvalues of F, i.e. $\lambda_1, ..., \lambda_k$, to be inside the unit circle. Imposing additional restrictions on the eigenvalues of F, an explicit form of the solution C of the Sylvester matrix equation in (17) can be obtained as follows:

Proposition 3 *Let* $F, A \in \mathbb{R}^{k \times k}$ *where* $\lambda_1, ..., \lambda_k$ *are the eigenvalues of* F *and* $\mu_1, ..., \mu_k$ *are the eigenvalues of* A. *Then, for any matrix* $H \in \mathbb{R}^{k \times k}$, *it follows that the companion Sylvester matrix equation in (17) has*

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a unique solution whenever $\lambda_i \mu_j < 1$ for all $i, j = 1, ..., k$ and this solution is given by:

$$C = -\sum_{s=0}^{\infty} F^s H A^s. \tag{25}$$

Proof. Let me define the following recursion: $FC_{r-1}A - C_r = H$ for iterations $r = 1, 2, 3, ...$ with the initial condition $C_0 = 0_k$ where 0_k is a $k \times k$ matrix of zeros. If this recursion converges as r goes to infinity, then by construction the limit characterizes the solution of the Sylvester matrix equation, i.e. $\lim_{r \to \infty} C_r = -\sum_{s=0}^{\infty} F^s H A^s = C$. The convergence condition is equivalent to $\lim_{s \to \infty} F^s H A^s = 0$. It naturally follows that any eigenvalue of $F^s H A^s$ must be proportional to the $s-$power of the product between the eigenvalues of F and A, i.e. $(\lambda_i \mu_i)^s$ for any $i = 1, ..., k$. Hence, if all cross products between the eigenvalues of F and A are strictly less than one, the corresponding eigenvalues for $F^s H A^s$ must go to zero in the limit as $s \to \infty$ and this suffices to show that the recursion must converge as stated (Lancaster and Tismenetsky (1985, Chapter 12.3)). ■

Proposition 3 implies that whenever the product of the spectral radii of matrices A and F is strictly less than one, a unique solution C exists that takes the special form of an infinite sum. Furthermore, this special case permits the straightforward computation of the solution to the companion Sylvester matrix equation via a recursion on a convergent sequence as suggested by the proof of the proposition. I explore later on a related numerical algorithm and other alternative efficient algorithms to compute the unique solution of the Sylvester matrix equation in (17). Often a numerical solution rather than one in closed-form form is all that is needed to recover the solution to the canonical forward-looking part of the LRE model.

Even when a solution C to the companion Sylvester matrix equation in (17) exists and is unique, characterizing the solution of the canonical forward-looking part of the LRE model with a finite-order VAR form requires C also to be nonsingular. If C can be shown to be nonsingular, its inverse C^{-1} can be computed in order to obtain the finite-order VAR representation given by Lemma 1 and Corollary 1. Proving the existence of C and its uniqueness does not suffice to ensure C is also invertible. For some $H \in \mathbb{R}^{k \times k}$, the unique solution C that exists can be singular. For example, it follows—since $vec(C) = \mathcal{A}^{-1} vec(H)$ and $\mathcal{A} := [(A^T \otimes F) - I_{k^2}]$ is nonsingular and invertible (Proposition 1)—that the unique solution is $C = 0_k$ whenever $H = 0_k$ and this solution is clearly singular.

The following condition must hold in order to ensure that C is invertible:

Condition 2 *Assume the conditions stated in Proposition 1 and Proposition 2 on $F, A \in \mathbb{R}^{k \times k}$ are satisfied and a unique solution C exists for the companion Sylvester matrix equation in (17). Then, for a given matrix $H \in \mathbb{R}^{k \times k}$, the solution C is said to be nonsingular and invertible if and only if C has full rank. In other words, if and only if $rank(C) = k$.*

Condition 2 is straightforward and follows directly from the terms of the invertible matrix theorem (Strang (2016)). Related to this rank condition, there are additional results that tie the properties of the matrices $F, A, H \in \mathbb{R}^{k \times k}$ that underpin the Sylvester matrix equation to a rank minimization condition. By Roth's removal theorem (Lancaster and Tismenetsky (1985, Chapter 12.5)), the Sylvester matrix equation in (17) has a solution $C \in \mathbb{R}^{k \times k}$ if and only if there exists a nonsingular matrix $P \in \mathbb{R}^{k \times k}$ such that:

$$P \begin{pmatrix} F & HA^{-1} \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & A^{-1} \end{pmatrix} P. \tag{26}$$

Then, the following rank identity has been noted elsewhere (Lin and Wimmer (2011)):

$$\min \left\{ rank \left(FC - CA^{-1} - HA^{-1} \right) \mid C \in \mathbb{R}^{k \times k} \right\}$$

$$= \min \left\{ rank \left[P \begin{pmatrix} F & HA^{-1} \\ 0 & A^{-1} \end{pmatrix} - \begin{pmatrix} F & 0 \\ 0 & A^{-1} \end{pmatrix} P \right] \mid \forall P \in \mathbb{R}^{k \times k} \text{ s.t. } rank \left(P \right) = k \right\}. \qquad (27)$$

These and related results in stability and control theory provide ways to connect the properties of the matrices $F, A, H \in \mathbb{R}^{k \times k}$ to the rank condition on the invertibility of the unique solution C to the Sylvester matrix equation (Condition 2).

I leave for future research the full exploration of those connections. The reason for this is purely practical. If a solution exists and is unique according to the conditions stated in Proposition 1 and Proposition 2, then computing the matrix C is all that is needed to describe the solution to the canonical forward-looking part of the LRE model in state-space form as given by equations $(8) - (9)$ under conditions $(12) - (13)$. Then, it is straightforward to check the rank condition for the invertibility of the computed matrix C (Condition 2). If that rank condition is violated, then one can conclude that the solution to the canonical forward-looking part of the LRE model does not admit a finite-order VAR representation. If that rank condition is satisfied, it follows that a finite-order VAR representation exists given by equation (16) as indicated in Lemma 1 (and (19) from Corollary 1). In that case, the VAR specification also permits the recovery of the true economic shocks underlying the model from the observed data—fundamentalness holds (Corollary 2).

3.2 Numerical Methods and Algorithms

It follows from Proposition 1 that a unique solution C of the companion Sylvester matrix equation in (17) exists if and only if the $k^2 \times k^2$ square matrix $\mathcal{A}$ is invertible. If so, the unique solution takes the form of a linear system with k^2 equations and k^2 unknown variables given by $vec\left(C\right) = \mathcal{A}^{-1} vec\left(H\right)$ which can be solved in $O\left(k^6\right)$ operations. Although obtaining the solution in this way is straightforward, there are algorithms and methods that can improve efficiency in the numerical computation of C. I base my approach on three steps: First, a linear transformation of the companion Sylvester matrix equation based on Schur's triangulation. Second, the transformed equation is solved (given that Schur's triangulation permits a recursive implementation). And, finally, the inverse transformation of Schur's triangulation is applied to recover the solution to the original form of the companion Sylvester matrix equation.

Step 1. The first step of the approach is to implement the generalized Schur triangulation to re-write the companion Sylvester matrix equation in (17). I find the real Schur decompositions $F = UKU^T$ and $A = VQV^T$ where $U, V \in \mathbb{R}^{k \times k}$ are unitary matrices of dimension k such that $UU^T = U^TU = I_k$ and $VV^T = V^TV = I_k$. The matrices $K, Q \in \mathbb{R}^{k \times k}$, referred to as the Schur forms corresponding to $F, A \in \mathbb{R}^{k \times k}$ respectively, are both upper triangular.[11] The eigenvalues of F and A are then the diagonal entries of the (upper) triangular matrices K and Q, respectively. Hence, I can re-write the companion Sylvester matrix equation—i.e. $FCA - C = H$ with $F, A, H \in \mathbb{R}^{k \times k}$ given and $C \in \mathbb{R}^{k \times k}$ to be determined—as follows:

$$KYQ - Y = R, \qquad (28)$$

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[11] Since K is similar to F, they both have the same eigenvalues. The same way, since Q is similar to A, their eigenvalues are the same.

where $Y = U^T C V$ and $R = U^T H V$.

Step 2. The second step of the approach is to solve the transformed Sylvester matrix equation in (17). The transformed equation can be vectorized to obtain:

$$\overline{\mathcal{A}} := \left[\left(Q^T \otimes K \right) - I_{k^2} \right], \tag{29}$$

$$\overline{\mathcal{A}} vec\left(Y\right) = vec\left(R\right). \tag{30}$$

Then, this can be solved directly by calculating the inverse of $\overline{\mathcal{A}}$ and using standard matrix algebra to solve the linear system $vec\left(Y\right) = \overline{\mathcal{A}}^{-1} vec\left(R\right)$ for Y.

Step 3. The last step of the approach is to recover the solution C to the Sylvester matrix equation. For that, I simply undo Schur's triangulation as follows $C = UYV^T$.

Recursive Implementation of the Proposed Solution Method. Although the three-step approach laid out before works in general, the solution to the transformed system in $(29)-(30)$ can be further optimized under additional assumptions on the matrix F and, particularly, on the matrix A.

(a) The solution to the transformed Sylvester matrix equation given by $(29) - (30)$ permits a more efficient recursive implementation, if A is diagonalizable. The diagonalization theorem indicates that the $k \times k$ matrix A is diagonalizable if and only if A has k linearly independent eigenvectors (Strang (2016)). If A is diagonalizable, then the matrix S of its eigenvectors is invertible and $S^{-1}AS = M = diag\left(\mu_1, ..., \mu_k\right)$ is the diagonal matrix of its eigenvalues. A sufficient (but not necessary) condition for A to be diagonalizable is that all its k eigenvalues be distinct.[12] By the principal axis theorem, it follows that if A is a real matrix (i.e. all entries of A are real numbers) and symmetric (i.e., $A^T = A$), then A is diagonalizable as well (Strang (2016)). Hence, the additional assumption that A be a diagonalizable matrix does not appear to be too restrictive for most practical applications—given that the stochastic process for the forcing variables is often assumed symmetric—i.e. A is often posited as a real symmetric square matrix—and, even when the symmetry assumption is relaxed, generally the eigenvalues appear as distinct.

Assuming from now on that the matrix A is diagonalizable, I can re-write the companion Sylvester matrix equation—i.e. $FCA - C = H$ with $F, A, H \in \mathbb{R}^{k \times k}$ given and $C \in \mathbb{R}^{k \times k}$ to be determined—with the Schur triangulation of F as before but using the diagonalization of A to obtain:

$$K\widehat{Y}M - \widehat{Y} = \widehat{R}, \tag{31}$$

where $\widehat{Y} = U^T C S$ and $\widehat{R} = U^T H S$. Then, the transformed Sylvester matrix equation can be vectorized as:

$$\widehat{\mathcal{A}} := \left[\left(M^T \otimes K \right) - I_{k^2} \right], \tag{32}$$

$$\widehat{\mathcal{A}} vec\left(\widehat{Y}\right) = vec\left(\widehat{R}\right). \tag{33}$$

The matrices M^T and I_{k^2} are diagonal, while K is an upper triangular matrix. As a result, it follows that $\widehat{\mathcal{A}}$ itself must be an upper triangular matrix. The resulting linear system can be solved directly by calculating the inverse of $\widehat{\mathcal{A}}$ and using standard matrix algebra to solve $vec\left(\widehat{Y}\right) = \overline{\mathcal{A}}^{-1} vec\left(\widehat{R}\right)$ for the

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[12] A matrix A can be diagonalizable, yet have repeated eigenvalues. For example, the identity matrix I_k is diagonal (hence diagonalizable), but has one eigenvalue repeated k times (i.e., $\mu_i = 1$ for all $i = 1, ..., k$).

vector of unknowns $\widehat{Y}$. The inverse of an upper triangular matrix is also upper triangular.

Hence, the diagonalization of A can help reduce the number of calculations needed to compute the solution to the transformed Sylvester matrix equation. Moreover, the resulting linear system lends itself to a recursive implementation that does not require the computation of the inverse of $\widehat{\mathcal{A}}$ explicitly. Let me define $\widehat{\mathcal{A}} = [\widehat{a}_{i,j}]_{i,j=1}^{k^2} \in \mathbb{R}^{k^2 \times k^2}$ as well as the column-vectors $vec\left(\widehat{Y}\right) = [\widehat{y}_i]_{i=1}^{k^2}$ and $vec\left(\widehat{R}\right) = [\widehat{r}_i]_{i=1}^{k^2}$. Then, for any given $j = 1, ..., k^2$, it holds true that $\widehat{a}_{i,j} = 0$ for all $i = 1, ..., k^2$ and $i < j$. It follows from here that $\widehat{a}_{k^2,k^2}\widehat{y}_{k^2} = \widehat{r}_{k^2}$ pins down $\widehat{y}_{k^2}$. Given $\widehat{y}_{k^2}$, the expression for $\widehat{y}_{k^2-1}$ can be immediately obtained from $\widehat{a}_{k^2-1,k^2-1}\widehat{y}_{k^2-1} + \widehat{a}_{k^2-1,k^2}\widehat{y}_{k^2} = \widehat{r}_{k^2-1}$. Given $\widehat{y}_{k^2}$ and $\widehat{y}_{k^2-1}$, the expression for $\widehat{y}_{k^2-2}$ is derived from $\widehat{a}_{k^2-2,k^2-2}\widehat{y}_{k^2-2} + \widehat{a}_{k^2-2,k^2-1}\widehat{y}_{k^2-1} + \widehat{a}_{k^2-2,k^2}\widehat{y}_{k^2} = \widehat{r}_{k^2-2}$. And so on and so forth. Then, once the matrix $\widehat{Y}$ is completed in this recursive way, the last step of the procedure is to recover the solution C to the Sylvester matrix equation. For that, I simply undo the transformation as follows $C = U\widehat{Y}S^{-1}$.

(b) The computation of matrix C can be further improved whenever F and A are both diagonalizable matrices. The diagonalization theorem implies that the $k \times k$ square matrices F and A are diagonalizable if and only if each of these matrices has k linearly independent eigenvectors, i.e. if and only if the rank of the matrix formed by the eigenvectors is k. I also know that if both matrices are real symmetric, they would be diagonalizable. Furthermore, if the eigenvalues of each matrix are distinct, this is sufficient (albeit not necessary) for each matrix to be diagonalizable as well. Assuming matrices F and A can be diagonalized— i.e., using $F = T\Lambda T^{-1}$ and $A = SMS^{-1}$ where $\Lambda = diag\left(\lambda_1, ..., \lambda_k\right)$ and $M = diag\left(\mu_1, ..., \mu_k\right)$—I obtain the following transformation of the companion Sylvester matrix equation:

$$\left(T\Lambda T^{-1}\right) C \left(SMS^{-1}\right) - C = H. \tag{34}$$

Multiplying the left-hand side of this matrix equation by T^{-1} and the right-hand side by S, it follows that:

$$\Lambda T^{-1}CSM - T^{-1}CS = T^{-1}HS. \tag{35}$$

Let $\widetilde{Y} = T^{-1}CS$ and $\widetilde{R} = T^{-1}HS$. Then, it find that:

$$\Lambda\widetilde{Y}M - \widetilde{Y} = \widetilde{R}. \tag{36}$$

Denoting the (i,j)-th entry of $\widetilde{Y}$ as $\widetilde{y}_{ij}$ and the (i,j)-th entry of $\widetilde{R}$ as $\widetilde{r}_{ij}$, the diagonalized Sylvester matrix equation can be rewritten simple as:

$$\lambda_i\mu_j\widetilde{y}_{ij} - \widetilde{y}_{ij} = \widetilde{r}_{ij}, \; \forall i,j = 1, ..., k, \tag{37}$$

which means that:

$$\widetilde{y}_{ij} = \frac{\widetilde{r}_{ij}}{\lambda_i\mu_j - 1}. \tag{38}$$

Since the eigenvectors and eigenvalues of a diagonalizable matrix can be found with only $O\left(k^3\right)$ operations, the transformed Sylvester matrix equation can be solved more efficiently in this way. Then, the matrix C can be immediately recovered undoing the transformation as $C = T\widetilde{Y}S^{-1}$.

Other Numerical Algorithms to Solve the Sylvester Equation. **(a) Bartels-Stewart Approach.**
A classical numerical algorithm for solving the Sylvester matrix equation is the Bartels-Stewart algorithm which makes use of the Schur decompositions of F and G to obtain a more efficient algorithm to compute the solution C (Bartels and Stewart (1972)). Using a Schur decomposition as before, the companion Sylvester matrix equation—i.e. $FCA - C = H$ with $F, A, H \in \mathbb{R}^{k \times k}$ given and $C \in \mathbb{R}^{k \times k}$ to be determined—can be re-written as in equation (28). Let Q_{ij} denote a block of the upper triangular matrix Q, and let Y and R be partitioned according to a column partitioning of Q. The key step is to exploit these facts to decompose the transformed Sylvester matrix equation in (28) into smaller Sylvester matrix equations by blocks as follows:

$$KY_1Q_{11} - Y_1 = R_1, \tag{39}$$

$$KY_jQ_{jj} - Y_j = R_j - K\sum\nolimits_{i=1}^{j-1} Y_jQ_{ij}, \ \forall j = 2, ..., k. \tag{40}$$

Each of the block equations in $(39) - (40)$ takes the form of the transformed Sylvester matrix equation in (28) given that the sum that appears on the right-hand side of (40) is recursively known, as indicated above.[13]

An improved modification of the Bartels-Stewart algorithm, known as the Hessenberg-Schur algorithm, was proposed in Golub *et al.* (1979). This algorithm uses the Hessenberg decomposition instead of the Schur decomposition to transform the companion Sylvester matrix equation (Golub and van Loan (1996), Anderson *et al.* (1996)). The Hessenberg decomposition implies $F = UKU^H$ and $A = VQV^H$ where $U, V \in \mathbb{R}^{k \times k}$ are unitary matrices of dimension k, U^H, V^H denote the corresponding conjugate transpose of those matrices and $K, Q \in \mathbb{R}^{k \times k}$ are the corresponding Hessenberg matrices.

The Sylvester matrix equation is a special case of the Lyapunov equation. Hence, the *dlyap* function in the Control Systems Toolbox in Matlab which solves the discrete-time Lyapunov equation can be used to solve the companion Sylvester matrix equation in (17) as follows: $C = dlyap(F, A, -H)$. This function uses the SLICOT (Subroutine Library In COntrol Theory) library—with a routine that implements the Hessenberg-Schur algorithm. Starting with R2014a, the matrix C can also be computed in Matlab via the *sylvester* function as $C = sylvester(F, -inv(A), H * inv(A))$.[14]

(b) Doubling Algorithm. The doubling algorithm exploits the convergence result posited in Proposition 3 which establishes that, for any matrix $H \in \mathbb{R}^{k \times k}$, the companion Sylvester matrix equation in (17) has a unique exact solution given by (25) whenever $\lambda_i\mu_j < 1$ for all $i, j = 1, ..., k$ where $\lambda_1, ..., \lambda_k$ are the eigenvalues of F and $\mu_1, ..., \mu_k$ are the eigenvalues of A. The doubling algorithm defines the following sequence:

$$\begin{aligned} \Lambda_{s+1} &= \Lambda_s\Lambda_s, \\ \Psi_{s+1} &= \Psi_s\Psi_s, \\ C_{s+1} &= C_s + \Lambda_s C_s \Psi_s, \end{aligned} \tag{41}$$

where $\Lambda_0 = F$, $\Psi_0 = A$, and $C_0 = -H$. This sequence converges to the solution of the companion Sylvester equation C as $s \to \infty$. By repeated substitution, it can be shown that each iteration doubles the number of

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[13]If Q is a real matrix from the Schur decomposition, then Q_{jj} for all $j = 1, 2, ..., k$ must be either a scalar or a 2×2 matrix.
[14]Further details on standard implementation methods using Matlab can be found in Sima and Benner (2015). For further references on the Matlab function *dlyap*, see: http://www.mathworks.com/help/control/ref/dlyap.html and http://slicot.org/matlab-toolboxes/basic-control/basic-control-fortran-subroutines. For reference on the Matlab function *sylvester*, see: http://www.mathworks.com/help/matlab/ref/sylvester.html

terms in the sum—hence the name of the algorithm—such that:

$$C_r = -\sum_{s=0}^{2^r-1} F^s H A^s, \tag{42}$$

becomes arbitrarily close to the solution $C = -\sum_{s=0}^{\infty} F^s H A^s$ as r gets arbitrarily large. Further discussion on this popular algorithm and Matlab codes to implement it can be found, among others, in Anderson *et al.* (1996).

4 An Application to the Workhorse New Keynesian Model

A Univariate Model of Inflation: The Hybrid Phillips Curve. The hybrid Phillips curve with backward- and forward-looking components, arising from the well-known Calvo (1983)-type model of price-setting behavior with indexation developed by Yun (1996), features prominently in the New Keynesian literature. The hybrid Phillips curve can be specified generically as:

$$\pi_t = \gamma_f \mathbb{E}_t \left(\pi_{t+1} \right) + \gamma_b \pi_{t-1} + e_t, \tag{43}$$

where π_t is the inflation rate, and $\mathbb{E}_t \left(\pi_{t+1} \right)$ is the expected inflation rate next period. The parameters $\gamma_f, \gamma_b > 0$ determine the sensitivity of current inflation to inflation expectations (the forward-looking part) and lagged inflation (the backward-looking part) and satisfy that $\gamma_f + \gamma_b \leq 1$. The variable e_t refers to the exogenous real marginal cost which is assumed to evolve according to a given first-order autoregressive process, i.e.,

$$e_t = \eta e_{t-1} + \sigma_\delta \delta_t, \tag{44}$$

where δ_t is i.i.d. white noise with mean zero and variance of one. The persistence parameter $-1 < \eta < 1$ is expected to be less than one in absolute value to ensure the stationarity of the process, while the parameter $\sigma_\delta > 0$ pins down the volatility of the real marginal cost shock.

The simple inflation model given by the system in $(43) - (44)$ consists of just one endogenous variable, π_t, and one forcing variable, e_t. Hence, it is not difficult to obtain a closed-form solution for inflation in this case and to characterize it in autoregressive form analytically. Using the notation introduced in Section 2, the model-implied relationship between the vector of endogenous variables $W_t = (\pi_t)$ and the vector of forcing variables $X_t = (e_t)$ can be cast in the first-order form of the LRE model given by $(1) - (2)$ with 1×1 composite matrices of the form $\Phi_1 = (\gamma_b)$, $\Phi_2 = (\gamma_f)$, $\Phi_3 = (1)$, $A = (\eta)$ and $B = (\sigma_\delta)$.

To start, I consider a system reduction to split the solution of the model given in $(43) - (44)$ into a backward-looking part and a forward-looking part. From the quadratic matrix equation (5) in Condition 1 applied to this example, I find that the decoupling depends on the roots of the following characteristic equation:

$$\theta^2 - \frac{1}{\gamma_f}\theta + \frac{\gamma_b}{\gamma_f} = 0, \tag{45}$$

i.e., $\theta_1 = \frac{1 - \sqrt[2]{1 - 4\gamma_f \gamma_b}}{2\gamma_f}$ and $\theta_2 = \frac{1 + \sqrt[2]{1 - 4\gamma_f \gamma_b}}{2\gamma_f}$. The solution Θ that permits splitting the backward- and forward-looking parts of the model requires a stable eigenvalue that lies within the unit circle to exist, i.e. $\Theta = (\theta_1)$ if and only if $|\theta_1| < 1$. As can be easily seen, the existence of the solution Θ depends

solely on the parameters γ_f and γ_b. If such a solution exists, then the transformed endogenous variable $Z_t \equiv W_t - \Theta W_{t-1} = (\pi'_t)$ takes the following form: $\pi'_t = \pi_t - \theta_1 \pi_{t-1}$ where θ_1 is the corresponding stable (real-valued) root of the quadratic equation.

As indicated by equation (6) before, the forward-looking part of the hybrid Phillips curve-based model for $Z_t = (\pi'_t)$ becomes:

$$\Gamma_0 \pi'_t = \Gamma_1 \mathbb{E}_t \left[\pi'_{t+1} \right] + \Gamma_2 e_t, \tag{46}$$

where $\Gamma_0 \equiv (1 - \gamma_f \theta_1)$, $\Gamma_1 \equiv (\gamma_f)$, and $\Gamma_2 \equiv (1)$ are conforming 1×1 matrices and the forcing variable e_t remains untransformed. It follows from the properties of the roots of the quadratic equation that $1 - \gamma_f \theta_1 = \gamma_f \theta_2$. Hence, so long as θ_2 is different from zero, the 1×1 matrix Γ_0 is invertible and the canonical forward-looking part of the LRE model can be re-expressed as in equation (7), i.e.,

$$\pi'_t = F \mathbb{E}_t \left[\pi'_{t+1} \right] + G e_t, \tag{47}$$

where $F \equiv \left((\gamma_f \theta_2)^{-1} \gamma_f \right) = \left((\theta_2)^{-1} \right)$ and $G \equiv \left((\gamma_f \theta_2)^{-1} \right)$. From the Blanchard and Kahn (1980) conditions applied to the system in (44) and (47), it is straightforward to show that a solution to canonical LRE model exists and is unique if and only if $|\theta_2| > 1$.

All of this ultimately implies that the full-fledged LRE model in $(43) - (44)$ can be split into a backward- and a forward-looking part and solved uniquely if and only if the roots of the quadratic equation in (45) satisfy that $|\theta_1| < 1$ and $|\theta_2| > 1$. Then, given equation (17) of Lemma 1, the companion Sylvester matrix equation for this model is given by $FCA - C = H$ where $F = \left(\frac{1}{\theta_2} \right)$, $A = (\eta)$ and $H \equiv -GA = \left(-\left(\frac{\eta}{\gamma_f \theta_2} \right) \right)$. The solution to this equation gives $C = \left(\frac{1}{\gamma_f} \left(\frac{\eta}{\theta_2 - \eta} \right) \right)$ which is well-conditioned and invertible if and only if $\eta \neq 0$ and $\theta_2 \neq \eta$. Therefore, the closed-form solution of the canonical forward-looking part of the LRE model under rational expectations maps the shocks into the transformed endogenous variables as in equation (15) above and can be expressed as:

$$\pi'_t = \pi_t - \theta_1 \pi_{t-1} = CA^{-1} e_t, \tag{48}$$

$$CA^{-1} \equiv \left(\frac{1}{\gamma_f} \left(\frac{1}{\theta_2 - \eta} \right) \right), \tag{49}$$

which, together with the autoregressive process specification give in (44), fully describes the inflation dynamics implied by this univariate model based on the hybrid Phillips curve.

I can infer the dynamics of the transformed inflation rate in autoregressive form as in (16) given by:

$$(\pi_t - \theta_1 \pi_{t-1}) = CAC^{-1} (\pi_{t-1} - \theta_1 \pi_{t-2}) + CA^{-1} B \delta_t, \tag{50}$$

$$CAC^{-1} \equiv (\eta), \ CA^{-1} B \equiv \left(\frac{1}{\gamma_f} \left(\frac{1}{\theta_2 - \eta} \right) \sigma_\delta \right). \tag{51}$$

From an economic point of view, this solution highlights the importance of the backward-looking component of the hybrid Phillips curve to understand the dynamics of inflation. The persistence of the inflation process is not solely determined by the persistence of the exogenous real marginal cost shock, η, but it also depends on the root θ_1 which is a composite of the backward-looking and forward-looking coefficients of the hybrid

Phillips curve (that is, a composite of $\gamma_f > 0$ and $\gamma_b > 0$).[15] The closed-form solution of the univariate hybrid Phillips curve model in (50) shows that it is possible to characterize the solution to an LRE model in finite-order autoregressive form. That, in turn, permits the identification of the fundamental economic shock forcing inflation in the univariate model, δ_t.

A Bivariate Monetary Model of Inflation. The method proposed in this paper provides the tools to generalize the logic behind the result in (50) to a more general setting with more than one endogenous and one forcing variables. The approach suggested in the paper helps characterize a finite-order VAR solution for a large class of LRE models by solving a companion quadratic matrix equation and a companion Sylvester matrix equation—checking the existence, uniqueness, and invertibility of its solution.

I start augmenting the hybrid Phillips curve-based model given by (44) and (47) with the following variant of the Taylor (1993) rule with inertia to introduce a monetary policy rule explicitly in the determination of inflation:

$$i_t = \rho_i i_{t-1} + (1 - \rho_i) [\psi_\pi \pi_t] + m_t, \tag{52}$$

where the policy rate is denoted i_t and the policy inertia is modelled with the parameter $0 < \rho_i < 1$. The policy rule responds to deviations of inflation under the Taylor principle with the parameter ψ_π set to $\psi_\pi > 1$. Here, the associated monetary policy shock m_t follows an exogenously given first-order autoregressive process of the following form:

$$m_t = \rho_m m_{t-1} + \sigma_\xi \xi_t, \tag{53}$$

where ξ_t is assumed to be i.i.d. white noise with mean zero and variance of one, and also uncorrelated at all leads and lags with δ_t. The persistence parameter $-1 < \rho_m < 1$ is expected to be less than one in absolute value to ensure the stationarity of the process, while the parameter $\sigma_\xi > 0$ pins down the monetary shock volatility.

Let me define the vector of endogenous variables as $W_t = (\pi_t, i_t)^T$, the vector of forcing variables as $X_t = (e_t, m_t)^T$, and the vector of innovations as $\epsilon_t = (\delta_t, \xi_t)^T$. The bivariate monetary model of inflation given by (43) and (52) in matrix form, i.e.,

$$\begin{pmatrix} 1 & 0 \\ -(1-\rho_i)\psi_\pi & 1 \end{pmatrix} \begin{pmatrix} \pi_t \\ i_t \end{pmatrix} = \begin{pmatrix} \gamma_b & 0 \\ 0 & \rho_i \end{pmatrix} \begin{pmatrix} \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \begin{pmatrix} \gamma_f & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{E}_t [\pi_{t+1}] \\ \mathbb{E}_t [i_{t+1}] \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_t \\ m_t \end{pmatrix}, \tag{54}$$

can be expressed in the form of (1) as follows:

$$W_t = \Phi_1 W_{t-1} + \Phi_2 \mathbb{E}_t [W_{t+1}] + \Phi_3 X_t, \tag{55}$$

$$\Phi_1 = \begin{pmatrix} \gamma_b & 0 \\ \gamma_b (1-\rho_i)\psi_\pi & \rho_i \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \gamma_f & 0 \\ \gamma_f (1-\rho_i)\psi_\pi & 0 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} 1 & 0 \\ (1-\rho_i)\psi_\pi & 1 \end{pmatrix}. \tag{56}$$

The shock processes in (44) and (53) can be cast in the form indicated by the matrix equation (8) with

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[15] After reversing the transformation, the dynamics of inflation implied by the model can be easily represented with a second-order autoregressive process as in (19) given by: $\pi_t = (\theta_1 + \eta) \pi_{t-1} - \eta\theta_1 \pi_{t-2} + \frac{1}{\gamma_f} \left(\frac{1}{\theta_2 - \eta} \right) \sigma_\delta \delta_t$. The characteristic quadratic equation associated with this second-order autoregressive process is $\phi^2 - (\theta_1 + \eta) \phi - (-\eta\theta_1) = 0$, with roots given by $\phi_1 = \frac{1}{2} (\theta_1 + \eta) - \frac{1}{2} \sqrt[2]{(\theta_1 + \eta)^2 - 4\eta\theta_1}$ and $\phi_1 = \frac{1}{2} (\theta_1 + \eta) + \frac{1}{2} \sqrt[2]{(\theta_1 + \eta)^2 - 4\eta\theta_1}$. The properties of the roots of the quadratic equation imply that $\phi_1 + \phi_2 = (\theta_1 + \eta)$ and $\phi_1\phi_2 = \eta\theta_1$.

conforming matrices A and B given by:

$$A = \begin{pmatrix} \eta & 0 \\ 0 & \rho_m \end{pmatrix}, \quad B = \begin{pmatrix} \sigma_\delta & 0 \\ 0 & \sigma_\xi \end{pmatrix}. \tag{57}$$

This constitutes the first-order form of the bivariate monetary model of inflation (equations $(1) - (8)$).

In order to solve the full-fledged model, I split the backward- and forward-looking parts of the model as indicated in Condition 1. To solve the quadratic matrix equation in (5), I construct the following two companion matrices:

$$D = \begin{bmatrix} 1 & 0 & -\gamma_b & 0 \\ 0 & 1 & -\gamma_b \left(1 - \rho_i\right)\psi_\pi & -\rho_i \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} \gamma_f & 0 & 0 & 0 \\ \gamma_f \left(1 - \rho_i\right)\psi_\pi & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{58}$$

and solve the corresponding generalized eigenvalue problem (see the Appendix for technical details). As a result, I obtain the following ordered matrix of generalized eigenvalues Q and their associated matrix of eigenvectors V:

$$Q = \begin{pmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \rho_i & 0 & 0 \\ 0 & 0 & \theta_2 & 0 \\ 0 & 0 & 0 & \infty \end{pmatrix}, \quad V = \begin{pmatrix} \frac{\theta_1 - \rho_i}{(1-\rho_i)\psi_\pi} & 0 & \frac{\theta_2 - \rho_i}{(1-\rho_i)\psi_\pi} & 0 \\ \theta_1 & \rho_i & \theta_2 & 1 \\ \frac{\theta_1 - \rho_i}{(1-\rho_i)\psi_\pi \theta_1} & 0 & \frac{\theta_2 - \rho_i}{(1-\rho_i)\psi_\pi \theta_2} & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \tag{59}$$

where $\theta_1 = \frac{1 - \sqrt[2]{1 - 4\gamma_f \gamma_b}}{2\gamma_f}$ and $\theta_2 = \frac{1 + \sqrt[2]{1 - 4\gamma_f \gamma_b}}{2\gamma_f}$ are defined exactly as in the univariate case. The matrices Q and V are already ordered so that the two stable eigenvalues come first.

From here it follows that $Q^1 = \begin{pmatrix} \theta_1 & 0 \\ 0 & \rho_i \end{pmatrix}$ and $V^{21} = \begin{pmatrix} \frac{\theta_1 - \rho_i}{(1-\rho_i)\psi_\pi \theta_1} & 0 \\ 1 & 1 \end{pmatrix}$, so the companion quadratic matrix equation in (5) has the following solution:

$$\Theta = V^{21} Q^1 \left(V^{21}\right)^{-1} = \begin{pmatrix} \theta_1 & 0 \\ \left(1 - \rho_i\right)\psi_\pi \theta_1 & \rho_i \end{pmatrix}, \tag{60}$$

which is lower triangular. The solution Θ found in (60) permits splitting the backward- and forward-looking parts of the bivariate model of inflation. To do so, it requires two eigenvalues that are stable and lie within the unit circle to exist, i.e. it requires $|\theta_1| < 1$ and $|\rho_i| < 1$. Given that by construction I already assume that $0 < \rho_i < 1$, the solution Θ that I seek to characterize depends solely on whether the parameters γ_f and γ_b imply also that $|\theta_1| < 1$. If such a solution Θ exists, then the transformed endogenous variables $Z_t \equiv W_t - \Theta W_{t-1} = (\pi'_t, i'_t)$ take the following form: $\pi'_t = \pi_t - \theta_1 \pi_{t-1}$ where θ_1 is the same stable root as in the univariate case, and $i'_t = i_t - (1 - \rho_i)\psi_\pi \theta_1 \pi_{t-1} - \rho_i i_{t-1}$. The transformed short-term interest rate i'_t needs to be adjusted with its own lag as well as with lagged inflation, while the adjustment for the inflation variable π'_t is exactly the same as in the univariate case.

Then, the forward-looking part of the bivariate model of inflation can be expressed in the form of (6) as

follows:

$$\Gamma_0 Z_t = \Gamma_1 \mathbb{E}_t \left[Z_{t+1} \right] + \Gamma_2 X_t, \tag{61}$$

$$\Gamma_0 \equiv (I_k - \Phi_2 \Theta) = \begin{pmatrix} 1 - \gamma_f \theta_1 & 0 \\ -(1 - \rho_i) \psi_\pi \gamma_f \theta_1 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_f \theta_2 & 0 \\ -(1 - \rho_i) \psi_\pi \gamma_f \theta_1 & 1 \end{pmatrix}, \tag{62}$$

$$\Gamma_1 \equiv \Phi_2 = \begin{pmatrix} \gamma_f & 0 \\ \gamma_f (1 - \rho_i) \psi_\pi & 0 \end{pmatrix}, \; \Gamma_2 \equiv \Phi_3 = \begin{pmatrix} 1 & 0 \\ (1 - \rho_i) \psi_\pi & 1 \end{pmatrix}. \tag{63}$$

Whenever Γ_0 is nonsingular, the system of structural relationships for the forward-looking part of the bivariate monetary model of inflation implied by (6) can be expressed in the form of (7) as:[16]

$$Z_t = F \mathbb{E}_t \left[Z_{t+1} \right] + G X_t, \tag{64}$$

$$F \equiv (\Gamma_0)^{-1} \Gamma_1 = \begin{pmatrix} \frac{1}{\theta_2 \gamma_f} & 0 \\ \frac{\theta_1}{\theta_2} (1 - \rho_i) \psi_\pi & 1 \end{pmatrix} \begin{pmatrix} \gamma_f & 0 \\ \gamma_f (1 - \rho_i) \psi_\pi & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\theta_2} & 0 \\ (1 - \rho_i) \psi_\pi \left(\frac{1}{\theta_2} \right) & 0 \end{pmatrix}, \tag{65}$$

$$G \equiv (\Gamma_0)^{-1} \Gamma_2 = \begin{pmatrix} \frac{1}{\theta_2 \gamma_f} & 0 \\ \frac{\theta_1}{\theta_2} (1 - \rho_i) \psi_\pi & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (1 - \rho_i) \psi_\pi & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\theta_2 \gamma_f} & 0 \\ (1 - \rho_i) \psi_\pi \left(\frac{1}{\theta_2 \gamma_f} \right) & 1 \end{pmatrix}. \tag{66}$$

The solution of the bivariate model in state-space form includes matrices A and B in (57) to represent the stochastic dynamics of the forcing variables and the matrix equation in (9) to describe the mapping between the lagged forcing variables and their innovations into the transformed endogenous variables in the solution of the canonical LRE model. This, in turn, requires the conforming matrices C and D to satisfy the conditions given by $(12) - (13)$.

The matrices C and D are tied to the matrices F, H, and G that arise from the canonical form of the forward-looking part of the LRE model in (7) where F and H are given above in $(65) - (66)$ and[17]

$$H \equiv -GA = -\begin{pmatrix} \frac{1}{\theta_2 \gamma_f} & 0 \\ (1 - \rho_i) \psi_\pi \left(\frac{1}{\theta_2 \gamma_f} \right) & 1 \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \rho_m \end{pmatrix} = \begin{pmatrix} -\frac{\eta}{\theta_2 \gamma_f} & 0 \\ -\frac{\eta}{\theta_2} \frac{\psi_\pi}{\gamma_f} (1 - \rho_i) & -\rho_m \end{pmatrix}. \tag{67}$$

From Proposition 2, I check the existence and uniqueness of a solution C via the companion Sylvester matrix equation in (17). I compute the eigenvalues of F (that is, $\lambda_1 = \frac{1}{\theta_2}$, $\lambda_2 = 0$) and the eigenvalues of A (that is, $\mu_1 = \eta$, $\mu_2 = \rho_m$). Then, given that $\lambda_i \mu_j \neq 1$, for all $i, j = 1, 2$ if and only if $\theta_2 \neq \eta$ and $\theta_2 \neq \rho_m$, I conclude that a solution C to the companion Sylvester matrix equation exists and is unique.

A straightforward manipulation of the 2^2 equations implied by the Sylvester matrix equation leads me to characterize the conforming matrices C and D as follows:[18]

$$C = \begin{pmatrix} \frac{1}{\gamma_f} \left(\frac{1}{\theta_2 - \eta} \right) \eta & 0 \\ \psi_\pi (1 - \rho_i) \frac{1}{\gamma_f} \left(\frac{1}{\theta_2 - \eta} \right) \eta & \rho_m \end{pmatrix}, \; D \equiv CA^{-1}B = \begin{pmatrix} \frac{1}{\gamma_f} \left(\frac{1}{\theta_2 - \eta} \right) \sigma_\delta & 0 \\ \psi_\pi (1 - \rho_i) \frac{1}{\gamma_f} \left(\frac{1}{\theta_2 - \eta} \right) \sigma_\delta & \sigma_\xi \end{pmatrix}. \tag{68}$$

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[16] Notice that $\theta_1 + \theta_2 = \frac{1 - \sqrt[2]{1 - 4\gamma_f \gamma_b}}{2\gamma_f} + \frac{1 + \sqrt[2]{1 - 4\gamma_f \gamma_b}}{2\gamma_f} = \frac{1}{\gamma_f}$.

[17] Computing the eigenvalues of H (that is, $-\beta \frac{\eta}{\theta_2 \gamma_f}$ and $-\rho$), I find them to be non-zero if and only if $\eta \neq 0$ and $\rho \neq 0$ since $\theta_2 > 1$ and by assumption $\gamma_f, \gamma_f, \beta > 0$. Hence, the matrix H is nonsingular and invertible.

[18] Using the properties of vectorization and the Kronecker product, I write the companion Sylvester matrix equation in simple

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Checking Condition 2 is straightforward to see that $rank\,(C) = 2$ if and only if $\eta \neq 0$ and $\rho \neq 0$ since $\theta_2 > 1$ and by assumption $\gamma_f, \gamma_f > 0$ and $\psi_\pi > 1$ (the Taylor principle) must hold. All of this, in turn, implies that there exists a unique matrix C that solves the companion Sylvester matrix equation in (17) and is also invertible. Hence, the inverse of C is given as:

$$C^{-1} = \begin{pmatrix} \gamma_f\,(\theta_2 - \eta)\,\frac{1}{\eta} & 0 \\ -\psi_\pi\,(1 - \rho_i)\,\frac{1}{\rho_m} & \frac{1}{\rho_m} \end{pmatrix}. \tag{69}$$

Therefore, the forward-looking part of the bivariate inflation model has a VAR(1) representation in the form of (16) which can be expressed as:

$$Z_t = CAC^{-1}Z_{t-1} + CA^{-1}B\epsilon_t, \tag{70}$$

where

$$CAC^{-1} = \begin{pmatrix} \eta & 0 \\ (\eta - \rho_m)\,\psi_\pi\,(1 - \rho_i) & \rho_m \end{pmatrix}, \quad CA^{-1}B = \begin{pmatrix} \frac{1}{\gamma_f}\left(\frac{1}{\theta_2 - \eta}\right)\sigma_\delta & 0 \\ \psi_\pi\,(1 - \rho_i)\,\frac{1}{\gamma_f}\left(\frac{1}{\theta_2 - \eta}\right)\sigma_\delta & \sigma_\xi \end{pmatrix}. \tag{71}$$

Then, the finite-order VAR solution of the full-fledged LRE model in (19) becomes:

$$W_t = \Psi_1 W_{t-1} + \Psi_2 W_{t-2} + \Psi_3 \epsilon_t, \tag{72}$$

where

$$\Psi_1 \equiv \left(\Theta + CAC^{-1}\right) = \begin{pmatrix} \eta + \theta_1 & 0 \\ \psi_\pi\,(1 - \rho_i)\,(\eta + \theta_1 - \rho_m) & \rho_i + \rho_m \end{pmatrix}, \tag{73}$$

$$\Psi_2 \equiv -CAC^{-1}\Theta = \begin{pmatrix} -\eta\theta_1 & 0 \\ -\psi_\pi\,(1 - \rho_i)\,\eta\theta_1 & -\rho_i\rho_m \end{pmatrix}, \tag{74}$$

$$\Psi_3 \equiv CA^{-1}B = \begin{pmatrix} \frac{1}{\gamma_f}\left(\frac{1}{\theta_2 - \eta}\right)\sigma_\delta & 0 \\ \psi_\pi\,(1 - \rho_i)\,\frac{1}{\gamma_f}\left(\frac{1}{\theta_2 - \eta}\right)\sigma_\delta & \sigma_\xi \end{pmatrix}. \tag{75}$$

From an economic point of view, the solution of the bivariate monetary model of inflation presented here indicates that there are no spillovers from lagged interest rates into current inflation. Spillovers are only

linear form as $\mathcal{A}vec\,(C) = vec\,(H)$ where:

$$\mathcal{A} := \left[\left(A^T \otimes F\right) - I_{k^2}\right] = \begin{pmatrix} \frac{\eta}{\theta_2} - 1 & 0 & 0 & 0 \\ (1 - \rho_i)\,\psi_\pi\left(\frac{\eta}{\theta_2}\right) & -1 & 0 & 0 \\ 0 & 0 & \frac{\rho_m}{\theta_2} - 1 & 0 \\ 0 & 0 & (1 - \rho_i)\,\psi_\pi\left(\frac{\rho_m}{\theta_2}\right) & -1 \end{pmatrix}.$$

From here it follows that:

$$vec\,(C) = \mathcal{A}^{-1}vec\,(H) = \begin{pmatrix} \frac{\theta_2}{\eta - \theta_2} & 0 & 0 & 0 \\ \frac{\eta}{\eta - \theta_2}(1 - \rho_i)\,\psi_\pi & -1 & 0 & 0 \\ 0 & 0 & -\frac{\theta_2}{\theta_2 - \rho_m} & 0 \\ 0 & 0 & -\frac{\rho_m}{\theta_2 - \rho_m}(1 - \rho_i)\,\psi_\pi & -1 \end{pmatrix} \begin{pmatrix} -\frac{\eta}{\theta_2\gamma_f} \\ -\frac{\eta}{\theta_2}\frac{\psi_\pi}{\gamma_f}(1 - \rho_i) \\ 0 \\ -\rho_m \end{pmatrix} = \begin{pmatrix} \frac{\eta}{\gamma_f(\theta_2 - \eta)} \\ \frac{\eta}{\gamma_f(\theta_2 - \eta)}\psi_\pi\,(1 - \rho_i) \\ 0 \\ \rho_m \end{pmatrix}$$

Undoing the vectorization, the solution C in (68) follows.

from lagged inflation into the policy rate itself. The policy parameter ψ_π determines the magnitude of the spillovers from lagged inflation into the current policy rate—while the difference in persistence between non-monetary and monetary shocks implied by $(\eta + \theta_1 - \rho_m)$ influences the sign of the spillover from last period's inflation into the current policy rate. The policy parameter ψ_π plays a key role in explaining the contribution of the monetary policy shock innovation relative to that of the real marginal shock innovation (the non-monetary shock) in accounting for the policy rate volatility. However, current monetary policy shocks do not contribute to inflation fluctuations.

In other words, monetary policy has no effect on inflation determination in the bivariate LRE model given by (43), (44), (52) and (53). This is because the exogenous process for real marginal costs alone drives the dynamics of inflation via the hybrid Phillips curve. In fact, the solution of inflation is exactly the same as that of the univariate case and could have been derived separately since there are no linkages built into the model between the dynamics of inflation and the policy rate.

The Three-Variable New Keynesian Model of Inflation. A further extension of the inflation model that gives monetary policy a distinct role in the determination of real marginal costs and inflation is required. To do so, I follow the approach underlying the workhorse three-equation New Keynesian model which partly endogenizes the real marginal costs and connects them explicitly to monetary policy actions (similar to Carlstrom *et al.* (2009)). To be more precise, I retain the exogenous real marginal cost component e_t in the hybrid Phillips curve equation in (43) but augmenting the specification with an endogenous real marginal cost component that is proportional to the output gap y_t, i.e.,

$$\pi_t = \gamma_f \mathbb{E}_t \left(\pi_{t+1} \right) + \gamma_b \pi_{t-1} + \kappa y_t + e_t, \tag{76}$$

where the parameter $\kappa > 0$ identifies the slope of the hybrid Phillips curve. In line with the New Keynesian literature, I refer to the exogenous component e_t in this context as a cost-push shock (which implicitly bundles up a number of distinct shocks—potentially including TFP shocks, government expenditure shocks, etc., but not monetary policy shocks).

External habits (à la Campbell and Cochrane (1999)) lead the output gap y_t to evolve according to the following hybrid dynamic IS equation:

$$y_t = (1 - h) \mathbb{E}_t \left(y_{t+1} \right) + h y_{t-1} - \left(\frac{2h - 1}{\sigma} \right) \left(i_t - \mathbb{E}_t \left(\pi_{t+1} \right) - r_t^n \right), \tag{77}$$

where $\sigma > 0$ determines the intertemporal elasticity of substitution, and the coefficient $h \geq 0$ introduces external habit persistence in the specification. Needless to say, whenever $h = 0$ equation (77) collapses to the familiar time-separable dynamic IS equation. The natural rate r_t^n is assumed to follow an exogenously given first-order autoregressive process:

$$r_t^n = \vartheta r_{t-1}^n + \sigma_\zeta \zeta_t, \tag{78}$$

where ζ_t is assumed to be i.i.d. white noise with zero mean and variance of one, and uncorrelated at all leads and lags with δ_t and ξ_t. The persistence parameter $-1 < \vartheta < 1$ is expected to be less than one in absolute value to ensure the stationarity of the process, while the parameter $\sigma_\zeta > 0$ pins down the natural rate shock volatility. The natural rate of interest is defined within the model as the real interest rate that would prevail absent all nominal rigidities—therefore, it is implicitly a function of all non-monetary shocks

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in the background.

Finally, I also modify the monetary policy rule in (52) as follows:

$$i_t = \rho_i i_{t-1} + (1 - \rho_i) \left[\psi_\pi \pi_t + \psi_y y_t \right] + m_t, \tag{79}$$

to allow policy rule to respond to the endogenous output gap. The corresponding policy parameters satisfy that $\psi_\pi > 1$ and $\psi_y > 0$.[19] Notice here that, from the perspective of the New Keynesian model, only the monetary policy shocks have a structural economic interpretation while the cost-push shocks and the natural rate shocks can be viewed can be viewed as combinations of deeper structural shocks that cannot be independently recovered separately from each other using the three-variable, three-equation workhorse New Keynesian framework.

Let me define the vector of endogenous variables as $W_t = (y_t, \pi_t, i_t)^T$, the forcing variables as $X_t = (r_t^n, e_t, m_t)^T$, and the vector of innovations as $\epsilon_t = (\zeta_t, \delta_t, \xi_t)^T$. The forward-looking part of the three-equation New Keynesian model of inflation given by (76), (77), and (79) can be expressed as:

$$D_0 W_t = D_1 W_{t-1} + D_2 \mathbb{E}_t \left[W_{t+1} \right] + D_3 X_t, \tag{80}$$

$$D_0 = \begin{pmatrix} 1 & 0 & \left(\frac{2h-1}{\sigma} \right) \\ -\kappa & 1 & 0 \\ -(1-\rho_i)\psi_y & -(1-\rho_i)\psi_\pi & 1 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} h & 0 & 0 \\ 0 & \gamma_b & 0 \\ 0 & 0 & \rho_i \end{pmatrix}, \quad D_2 = \begin{pmatrix} (1-h) & \left(\frac{2h-1}{\sigma} \right) & 0 \\ 0 & \gamma_f & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} \left(\frac{2h-1}{\sigma} \right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{81}$$

and re-written, whenever D_0 is nonsingular, in the form of (1):

$$W_t = \Phi_1 W_{t-1} + \Phi_2 \mathbb{E}_t \left[W_{t+1} \right] + \Phi_3 X_t, \tag{82}$$

$$\Phi_1 \equiv (D_0)^{-1} D_1, \ \ \Phi_2 \equiv (D_0)^{-1} D_2, \ \ \Phi_3 \equiv (D_0)^{-1} D_3. \tag{83}$$

The shock processes in (44), (53), and (78) can be cast in the form indicated by the matrix equation (8) with conforming matrices A and B given by:

$$A = \begin{pmatrix} \vartheta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \rho_m \end{pmatrix}, \ B = \begin{pmatrix} \sigma_\zeta & 0 & 0 \\ 0 & \sigma_\delta & 0 \\ 0 & 0 & \sigma_\xi \end{pmatrix}. \tag{84}$$

Then, the solution of the three-equation New Keynesian model can be derived following the steps of the procedure proposed in this paper by solving a companion quadratic matrix equation and a companion Sylvester matrix equation.

In this case, I illustrate the solution of the full-fledged LRE model numerically taking advantage of the

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[19] While the output gap may not be observable, one can rewrite the New Keynesian model in terms of the observable output and a stochastic process for output potential. The output potential is driven by the same shock process as the natural rate, so the three-equation New Keynesian model can still be cast with three endogenous variables—all of them observable with output in place of the output gap—and three exogenous shock processes. A related illustration extending the workhorse New Keynesian model (based on observable output, inflation and the policy rate) to a two-country setting can be found in Duncan and Martínez-García (2015).

set of Matlab codes and functions that accompany the paper to implement the procedure.[20] First, let me assume the parameters of the New Keynesian model specified for this application take the conventional parameterization presented in Table 1:

Table 1. Parameterization of the Three-Equation New Keynesian Model		
Structural Parameter	Notation	Value
Forward-looking weight on Phillips curve	γ_f	0.7
Backward-looking weight on Phillips curve	γ_b	0.29
Slope of the Phillips curve	κ	0.5
Intertemporal elasticity of substitution	σ	1
External habit formation parameter	h	0.6
Monetary Policy		
Monetary policy inertia	ρ_i	0.85
Monetary policy response to inflation deviations	ψ_π	1.5
Monetary policy response to output gap deviations	ψ_y	0.5
Exogenous Shock Parameters		
Persistence of the natural interest rate shock	ϑ	0.95
Volatility of the natural interest rate shock	σ_ζ	1
Persistence of the cost-push shock	η	0.8
Volatility of the cost-push shock	σ_δ	2
Persistence of the monetary policy shock	ρ_m	0.3
Volatility of the monetary policy shock	σ_ξ	0.7

The parameter values are chosen to illustrate the qualitative features of the three-equation New Keynesian model and the performance of the algorithms for the solution of the companion quadratic matrix equation and the companion Sylvester matrix equation. I use a laptop with Intel(R) Core(TM) i7 with 2.7GHz, 4 cores and 32GB of installed memory (RAM). The wall-clock time elapsed in computing and reporting the numerical results given the defaults of the code is 0.150890 seconds (CPU time is: 0.1716 seconds). Using the iterative algorithm to compute the solution to the quadratic matrix equation and Matlab's own implementation of the Hessenberg-Schur algorithm for Lyapunov equations to speed up the computation, the elapsed time falls to 0.088264 seconds (CPU time is: 0.1248 seconds).

The code confirms that, given the parameterization of Table 1, the solution Θ to the companion quadratic matrix equation has its roots inside the unit circle. Furthermore, the code also reports that the solution C to the companion Sylvester matrix equation exists and is both unique and invertible. Therefore, a

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[20] In terms of implementation, the solution of the full-fledged LRE model requires solving a quadratic matrix equation as in (5) and a companion Sylvester matrix equation as in (17). In the univariate and bivariate illustrations presented before, the solution of the model can be achieved analytically with standard matrix algebra. However, the paper includes a straightforward Matlab code implementation to compute numerically the solution of the three-equation New Keynesian model. I use those codes in the rest of this section and make them available in my website: https://sites.google.com/site/emg07uw/. The codes can be downloaded directly using this link: https://sites.google.com/site/emg07uw/econfiles/LRE_model_solution.zip?attredirects=0.

Straightforward manipulations of those codes can be made to adapt them to other LRE models that can be cast in the first-order form investigated in this paper—users of the codes are asked to include a citation of this paper in their work. The Matlab programs and functions appear free of errors, however I do appreciate all feedback, suggestions or corrections that you may have. While users are free to copy, modify and use the code for their work, I do not assume any responsibility for any remaining errors or for how the codes may be used or misused by users other than myself.

straightforward implementation of the algorithm implies that the three-equation New Keynesian model has a finite-order VAR(2) representation given by (19) under the parameterization reported in Table 1 which takes the following form:

$$W_t = \Psi_1 W_{t-1} + \Psi_2 W_{t-2} + \Psi_3 \epsilon_t, \tag{85}$$

$$\Psi_1 = \begin{pmatrix} 1.7257 & -0.2344 & 0.0140 \\ 0.9303 & 0.6517 & 0.1666 \\ 0.3162 & 0.0616 & 1.1885 \end{pmatrix},$$

$$\Psi_2 = \begin{pmatrix} -0.7918 & 0.0567 & 0.0643 \\ -0.8523 & -0.1542 & 0.0417 \\ -0.2512 & -0.0305 & -0.2408 \end{pmatrix}, \tag{86}$$

$$\Psi_3 = \begin{pmatrix} 1.4816 & 0.0097 & -0.8349 \\ 3.3205 & 3.8860 & -1.7135 \\ 0.8582 & 0.8751 & 0.2518 \end{pmatrix},$$

where the corresponding coefficient matrices $\Psi_1 \equiv (\Theta + CAC^{-1})$, $\Psi_2 \equiv -CAC^{-1}\Theta$, and $\Psi_3 \equiv CA^{-1}B$ have non-zero entries everywhere—for an otherwise standard parameterization. This is in contrast with the bivariate monetary model discussed before where there were no spillovers from lagged interest rates to current inflation. Therefore, I find that partly endogenizing the real marginal costs with the output gap is key for monetary policy and monetary policy shocks to play a role in the determination of inflation in the New Keynesian model.

From an economic perspective, the solution of the three-equation New Keynesian model shows that there could be spillovers from lagged interest rates to inflation whenever the real marginal costs are partly endogenous and tied to short-term interest rate movements through a dynamic IS equation. Moreover, monetary policy shocks in the full-fledged New Keynesian model contribute to drive the inflation dynamics unlike in the bivariate version of the model presented earlier. Given that the finite-order VAR(2) solution of the workhorse New Keynesian model exists, one may be able to identify the structural monetary policy shock itself—as well as all other fundamental economic shocks forcing the economy—directly from the observable data. Furthermore, I find that empirical evidence which hinges upon Cholesky (and, in general, zero) restrictions should be interpreted carefully as it may not have a structural interpretation (as noted, among others, by Carlstrom *et al.* (2009)).

The impact of monetary policy on the volatility, cyclicality and persistence of the endogenous variables depends in nonlinear ways on the policy parameters, ψ_π and ψ_y, that describe the systematic part of the monetary policy rule. Here, I exploit the finite-order VAR(2) representation to compute the theoretical moments of the workhorse New Keynesian model for the benchmark parameterization of ψ_π and ψ_y, but also for increasingly higher values of the anti-inflation bias ψ_π. I summarize the key findings describing the

business cycle implications in Table 2 below:

Table 2. Key Business Cycle Moments at Different Degrees of Anti-Inflationary Bias				
Mean	$\psi_\pi = 1.5$	$\psi_\pi = 2$	$\psi_\pi = 3$	$\psi_\pi = 4$
Output Gap	0	0	0	0
Inflation	0	0	0	0
Policy Rate	0	0	0	0
Std. Deviation				
Output Gap	5.76	5.39	5.31	5.36
Inflation	8.42	6.12	4.29	3.48
Policy Rate	6.32	5.42	4.86	4.73
Contemporaneous Comovement				
Output Gap	1	1	1	1
Inflation	0.46	0.30	0.17	0.11
Policy Rate	0.13	0.03	−0.08	−0.14
First-Order Autocorrelation				
Output Gap	0.88	0.91	0.93	0.93
Inflation	0.69	0.64	0.57	0.52
Policy Rate	0.95	0.94	0.92	0.90

Note: This table summarizes the key theoretical business cycle moments of the three-equation workhorse New Keynesian model keeping the parameterization invariant as in Table 1 except for the policy parameter ψ_π.

As can be seen in Table 2, the monetary policy parameter ψ_π can have a major impact over the business cycle even though the shock processes remain invariant. Generally, a higher value of ψ_π signals a stronger anti-inflation commitment on the part of the monetary authority and is associated with significant declines in inflation and policy rate volatility as measured with the theoretical standard deviations—albeit not for the output gap. It is worth pointing out that higher values of the policy parameter lead to a weaker contemporaneous correlation between the output gap and inflation and to a reversal of the contemporaneous correlation between the output gap and the policy rate. I also find a significant weakening of the persistence of inflation as measured by its first-order autocorrelation.

All of this suggests that shifts in the patterns of endogenous volatility, cyclicality, and persistence do not necessarily reflect changes in the underlying shock process forcing the endogenous variables but can simply be the result of changes in monetary policy altering the monetary policy transmission mechanism itself. I illustrate those changes in the transmission mechanism by plotting in Figure 1 the corresponding one-standard deviation (theoretical) impulse response functions (IRFs) at different degrees of the anti-inflation bias, ψ_π.

I note that a higher anti-inflation bias stance on monetary policy ($\uparrow \psi_\pi$) tends to dampen the impact on the endogenous variables of natural rate and monetary policy shock innovations. However, while the impact on inflation of cost-push shock innovations also declines, a higher ψ_π tends to amplify the effect (at least in the initial quarters) of those same innovations in the output gap and the policy rate. The findings in Figure 1 show that the transmission mechanism of structural shocks—monetary policy shocks in particular—and their spillovers ultimately depend in nonlinear ways on the features of the prevailing monetary policy regime.

That is, they depend on the policy parameters of the Taylor (1993) rule in the workhorse New Keynesian model as well as on other deep structural parameters related to preferences, technology, etc.

Figure 1. Theoretical IRFs at Different Degrees of Anti-Inflationary Bias
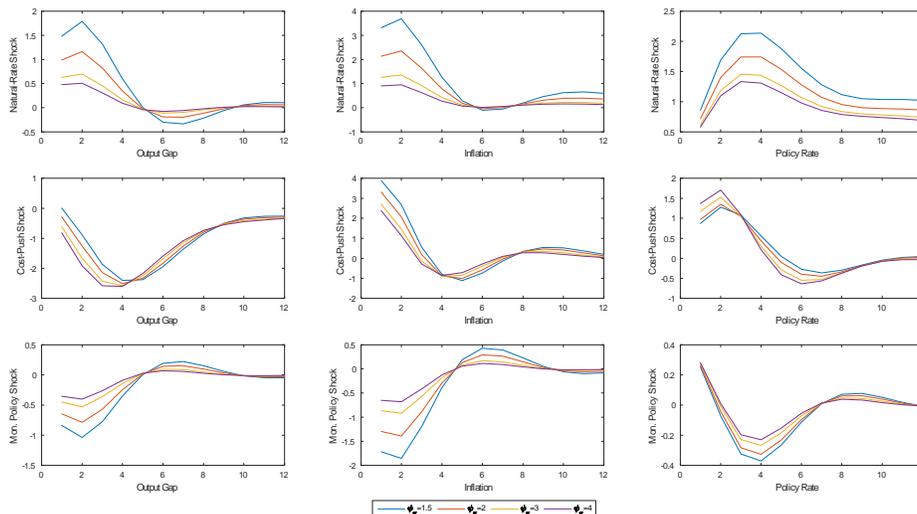
Note: This figure displays the theoretical impulse response functions (IRFs) of the three-equation workhorse New Keynesian model keeping the parameterization invariant as in Table 1 except for the policy parameter ψ_π.

The propagation of economic structural shocks is only part of the task, while exactly recovering them when fundamental—in particular, monetary policy shocks in the New Keynesian model—is crucial to understand their contribution to account for the observed data over the business cycle. Through the lens of the workhorse New Keynesian model parameterized as in Table 1, the findings in Figure 2 illustrate this point comparing the observed data on per-capita output growth, inflation, and the short-term policy rate against model-consistent simulations for each of the individual shocks separately (where these structural shocks are recovered from the observed data itself).

In other words, taking the observed data on per-capita output growth, inflation, and the short-term policy rate as given, I use equation (20) from Corollary 2 to recover the structural shocks for the three-equation New Keynesian model. Then, I simulate the endogenous variables feeding one of the recovered structural shocks at a time through the solution in (19) from Corollary 1 in order to assess each individual shock's contribution to account for the observed data. The comparison between these one-shock-only simulations and the actual observed data is what is plotted in Figure 2.

Mapping the endogenous variables of the New Keynesian model to the observed data requires some clarification. Most business cycle models abstract from movements in the labor force and assume output to be in per-capita terms—like the New Keynesian model does. Hence, the empirical counterpart to per-capita output in the New Keynesian model is calculated as the ratio of U.S. Real GDP in millions of chained 2009 Dollars (from the Bureau of Economic Analysis) over the U.S. civilian labor force 16-years and older in millions (from the Bureau of Labor Statistics).

The price index used is the quarterly U.S. Consumer Price Index (CPI) for all urban consumers–all items (from the Bureau of Labor Statistics). Finally, the short-term policy rate is constructed using the effective Federal Funds rate (% p.a.) (from the Federal Reserve Board) but replacing the observations while at the zero-lower bound with the Wu-Xia Shadow Federal Funds rate (as reported by the Federal Reserve Bank of Atlanta). All three series are reported at quarterly frequency. As customary, I transform the per-capita output and the price index taking logs to make the log-transformed series scale invariant.

The endogenous New Keynesian model variables are assumed to be stationary, so the filtering of their empirical counterparts has to take place outside of the model. A conventional way of getting the output gap by removing the trend out of log-per-capita output is to apply a first-difference filter to the series (multiplied by 400 to express this measure of per-capita output growth in percentage deviations and annualized). I consistently apply the same filter (multiplied by 400) to the price index as well to calculate a measure of inflation. Data in first differences still has non-zero mean average growth over the sample—accordingly, I demean the first-differenced log-per-capita output and the first-differenced price level. I also demean the short-term interest rate for the same reason.

Figure 2. Contribution of Each Individual Shock to the Observed U.S. Data
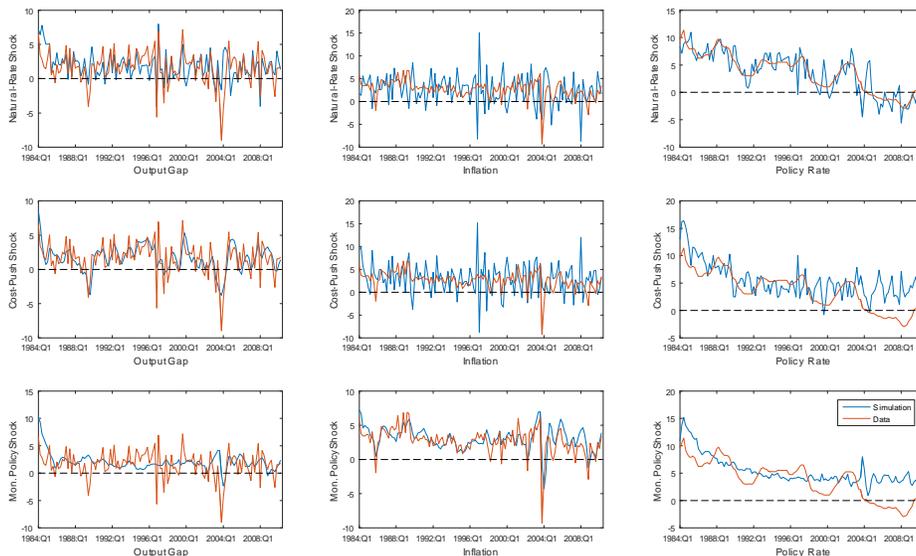
Note: This figure displays the observed data on the output gap, inflation and the short-term interest rate vs. model-consistent simulations of the corresponding variables based on the three-equation workhorse New Keynesian model keeping the parameterization invariant as in Table 1. The simulations correspond to the endogenous variables generated from the solution of the model feeding one of the structural shocks recovered at a time. The simulated series are modified with the addition of the corresponding sample mean of the corresponding observed variables for comparability.

The time series on this data starts in 1983:Q3 and ends in 2016:Q4 covering the entire Great Moderation period as well as the 2008 recession and its aftermath (with conventional Federal–Funds-based monetary policy stuck at the zero-lower bound). The sample mean is added to the model simulations in Figure 2

to make them comparable with the observed data that contains a non-zero sample mean.[21] The evidence reported shows—not surprisingly—that monetary policy shocks are key to understand inflation in the New Keynesian model. In turn, monetary shocks play a smaller role for per-capita output growth and for the short-term interest rate which appear largely driven by natural interest rate shocks and cost-push shocks, respectively. It is worth pointing out here that the New Keynesian model attributes the path of the policy rate in the aftermath of the 2008 recession (during the zero-lower bound episode) mostly to natural rate shocks rather than to cost-push shocks. This is consistent with a significant body of the theoretical literature that has built a narrative about the recent U.S. zero-lower bound experience partly around the perceived evolution (and decline) of the natural rate.

Hence, the method proposed in this paper to solve LRE models makes an important contribution to help us understand the transmission mechanism, the role of structural economic shocks and its mapping into finite-order VAR specifications for the workhorse New Keynesian model and for a large class of macro models that can be cast in the first-order form given by $(1) - (2)$.

5   Concluding Remarks

I propose a novel approach to solve a large class of LRE models. I represent the LRE model in its first-order form as a system of expectational first-order difference equations. Then, I reduce the first-order form by splitting the backward-looking and forward-looking parts of the model via the solution of a companion quadratic matrix equation. The forward-looking part is solved through the method of undetermined coefficients and describes the unique solution—when one exists—by means of a linear state-space form. Finally, I check the existence and uniqueness of a solution to the canonical (purely forward-looking part of the) LRE model and characterize its corresponding reduced-form solution in finite-order VAR form via a companion Sylvester matrix equation. With that at hand, I undo the transformation of variables achieved through system reduction to obtain the solution to the first-order LRE model.

An important contribution of the paper is the derivation of conditions under which the finite-order VAR solution of the canonical forward-looking part of the LRE model via the well-known Sylvester matrix equation is well-defined. The approach proposed in the paper not only provides a way to decouple the canonical form of the LRE model from its first-order specification and to characterize the corresponding solution in finite-order VAR form, but also checks its properties—existence, uniqueness and invertibility. Furthermore, the paper also makes a contribution to the existing literature on computational economics with the development of an integrated, unified algorithm to solve numerically for LRE models for which a unique finite-order VAR representation exists.

Solving LRE models with a VAR representation by this method is straightforward to implement, efficient, and can be handled easily with standard matrix algebra and conventional computational methods. The paper provides a number of functions for the solution of the companion quadratic matrix equation and the companion Sylvester matrix equation with an economically-relevant application to the study of the transmission mechanism of monetary policy in the context of the workhorse three-equation New Keynesian model. For this purpose, the paper provides an illustration of the procedure building up the workhorse New Keynesian model from the hybrid Phillips curve augmented to incorporate monetary policy and to

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[21] The sample mean of first-difference output per capita is 1.65% annualized, the sample average for first-differenced headline CPI is 2.67% annualized, and 3.76% is the period average for the nominal short-term interest rate.

endogenize the real marginal costs to open a feedback channel for monetary policy (and its shocks) to affect the determination of inflation.

Hence, the paper discusses the fundamental features of the New Keynesian model that permit the propagation of monetary policy shocks to inflation and the output gap—with particular attention to the role of the systematic part of the monetary policy rule (which determines the response to inflation and output deviations from their targets) and to the recovery of fundamental shocks (particularly the monetary policy shock). The paper also shows how under the conditions that ensure the existence of a finite-order VAR representation of the solution, the identification of fundamental shocks for empirical research—including the recovery of monetary shocks—can be made tractable.

Appendix

A A Closer Inspection of the LRE Model

A.1 The General Form of the LRE Model

The general form of a large class of multivariate LRE models can be written as

$$\Phi_{00} Y_t = \sum_{i=1}^{m} \Phi_{i0} Y_{t-i} + \sum_{i=0}^{m} \sum_{j=1}^{n} \Phi_{ij} \mathbb{E}_{t-i} \left(Y_{t+j-i} \right) + \Omega e_t, \tag{87}$$

where Y_t and e_t are the $r \times 1$ vectors of the endogenous and forcing variables of the LRE model, respectively. The vector Y_t collects the relevant subset r of endogenous variables and the r forcing variables of the LRE model, while the vector e_t are the innovations corresponding to the exogenous forcing variables. The matrices Φ_{ij} for all $i = 0, ..., m$ and $j = 0, ..., n$ and Ω are all $r \times r$ conforming matrices, and $\mathbb{E}_t (\cdot)$ represents the conditional expectations operator based on all current and lagged values of Y_t and e_t.[22] I assume that Φ_{00} in (87) is nonsingular and, without loss of generality, set it equal to the $r \times r$ identity matrix I_r.

As in Broze *et al.* (1985, 1990), the more general form given by (87) can be rewritten in more compact form as follows:

$$N_t = \Phi_1^* N_{t-1} + \Phi_2^* \mathbb{E}_t \left[N_{t+1} \right] + \Phi_3^* \overline{u}_t, \tag{88}$$

where

$$N_t = \left(U_t^T, U_{t-1}^T, ..., U_{t-m+1}^T \right)^T, \; U_t^T = \left(Y_t^T, \mathbb{E}_t \left(Y_{t+1}^T \right), ..., \mathbb{E}_t \left(Y_{t+n}^T \right) \right)^T,$$

$$\overline{u}_t = \left(u_t^T, 0_{l \times 1}^T, ..., 0_{l \times 1}^T \right)^T, \; u_t = \left(\left(\Omega e_t \right)^T, 0_{r \times 1}^T, ..., 0_{r \times 1}^T \right)^T,$$

$$\Phi_1^* = -D_0^{-1} D_1, \; \Phi_2^* = -D_0^{-1} D_1, \; \Phi_3^* = D_0^{-1}.$$

Here, $\overline{u}_t$ is a column-vector of dimension $q = m (n + 1) r$ and u_t is a column-vector of dimension $l = (n + 1) r$.

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[22] In case the processes have non-zero means, one should add a constant as well to the specification in (87).

The $q \times q$ square matrices D_i, $i = -1, 0, 1$ are defined as:

$$D_{-1} = \begin{pmatrix} \Psi_{-1} & 0_l & \dots & 0_l \\ 0_l & 0_l & \dots & 0_l \\ & & \ddots & \\ 0_l & 0_l & \dots & 0_l \end{pmatrix},$$

$$D_0 = \begin{pmatrix} \Psi_0 & \Psi_1 & \dots & \Psi_{m-1} \\ 0_l & I_l & \dots & 0_l \\ & & \ddots & \\ 0_l & 0_l & \dots & I_l \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0_l & 0_l & \dots & 0_l & \Psi_m \\ -I_l & 0_l & \dots & 0_l & 0_l \\ & & \ddots & & \\ 0_l & 0_l & \dots & -I_l & 0_l \end{pmatrix},$$

where Ψ_i, $i = -1, 0, ..., m$, are square matrices of dimension l given by:

$$\Psi_{-1} = \begin{pmatrix} 0_r & 0_r & \dots & 0_r & 0_r \\ -I_r & 0_r & \dots & 0_r & 0_r \\ & & \ddots & & \\ 0_r & 0_r & \dots & -I_r & 0_r \end{pmatrix},$$

$$\Psi_0 = \begin{pmatrix} I_r & -\Phi_{01} & \dots & -\Phi_{0n} \\ 0_r & I_r & \dots & 0_r \\ & & \ddots & \\ 0_r & 0_r & \dots & I_r \end{pmatrix},$$

$$\Psi_i = \begin{pmatrix} -\Phi_{i0} & -\Phi_{i1} & \dots & -\Phi_{in} \\ 0_r & 0_r & \dots & 0_r \\ & & \ddots & \\ 0_r & 0_r & \dots & 0_r \end{pmatrix} \quad \text{for } i = -1, 0, ..., m.$$

In the general form of the LRE model in (87), the endogenous variables are bundled together with the forcing variables in the vector Y_t. Naturally, the compact form derived in (88) inherits the same feature and the column-vector N_t of dimension $q = 2k$ combines k current and lagged forcing variables with k current and lagged endogenous variables (where both forcing and endogenous variables have the same finite lag order). Assume the vectors N_t and $\bar{u}_t$ are re-ordered such that $N_t = \left(W_t^T, X_t^T\right)^T$, where W_t is the vector of endogenous variables and X_t the vector of the forcing variables, and $\bar{u}_t$ follows the same consistent order. Then, the compact form in (88) can be rewritten as in the expectational difference system $(1) - (2)$ introduced in Section 2.

The compact system $(1) - (2)$ used in the paper simply presents the compact solution in two block sub-systems splitting the endogenous variables W_t which can be both forward-looking and backward-looking

from the exogenous forcing variables X_t which are only backward-looking but stochastic. Then, the compact system $(1) - (2)$ can be transformed into its canonical purely-forward looking form in (6) and (2) decoupling the forward-looking and the backward-looking terms of the endogenous variables, as explained in Section 2.1. Whenever $\Gamma_0 \equiv (I_k - \Phi_2 \Theta)$ is nonsingular, the canonical system of structural relationships for the endogenous variables implied by (6) can be rewritten as in (7). The resulting system of expectational difference equations in terms of transformed variables contains only forward-looking—and not backward-looking—terms.

A.2 The MA Representation of the LRE Solution

The canonical state-space representation of the large class of LRE models that I investigate in this paper can be given as in $(8) - (9)$, as noted in Section 2.2. With the assumption that A, B, C, and D are all conforming $k \times k$ matrices, it follows from equation (8) that $[I_k - AL] X_t = B\epsilon_t$ where I_k is a conforming identity matrix of dimension k, and L is the lag operator. If the eigenvalues of A are less than one in modulus, the solution of the LRE model in state-space form has an MA representation. In that case, X_t becomes a square summable polynomial given by $X_t = \sum_{j=0}^{\infty} [A]^j B\epsilon_{t-j}$. This expression can be shifted one period back and replaced in (9) to then obtain $Z_t = C \sum_{j=0}^{\infty} [A]^j B\epsilon_{t-1-j} + D\epsilon_t$.

The MA representation implied by the state-space solution in $(8) - (9)$ can be inverted under a simple condition and, therefore, represented with a VAR (possibly a VAR(∞)). Whenever the matrix D is nonsingular, replacing the vector ϵ_t in (8) using (9) gives:

$$\left[I_k - \left(A - BD^{-1}C \right) L \right] X_t = BD^{-1}Z_t. \tag{89}$$

If the eigenvalues of $A - BD^{-1}C$ are strictly less than one in modulus (the 'poor man's invertibility condition' of Fernández-Villaverde *et al.* (2007)), the inverse of the operator on the left-hand side of (89) gives X_t as a square summable polynomial in L satisfying:

$$X_t = \sum_{j=0}^{\infty} \left[A - BD^{-1}C \right]^j BD^{-1}Z_{t-j}. \tag{90}$$

Shifting this expression back one period and replacing it in (9), I obtain that:

$$Z_t = C \sum_{j=0}^{\infty} \left[A - BD^{-1}C \right]^j BD^{-1}Z_{t-1-j} + D\epsilon_t. \tag{91}$$

In general, the VAR(∞) representation of the solution to the canonical LRE model for the transformed endogenous variables Z_t can be reasonably well-approximated by a finite-order structural VAR model as shown in Inoue and Kilian (2002). However, I show in this paper that the finite-order VAR representation can be a unique and exact characterization of the canonical LRE model solution under certain conditions.

My methodological contribution is twofold: First, I propose a new approach to derive the state-space solution in $(8) - (9)$ from the solution of a companion Sylvester equation; Second, I derive a testable condition under which a finite-order VAR representation will be exact rather than an approximation for the canonical LRE model solution. The method is very efficient at computing the solution to a large class of LRE models. It involves checking a straightforward condition—the invertibility of the solution to the companion Sylvester equation—while simultaneously deriving the corresponding finite-order VAR representation of the solution

to a canonical LRE model of appropriately transformed variables (when one such solution exists).

B The Companion Quadratic Matrix Equation: A Closed-Form Solution

The quadratic matrix equation in (5) can be solved for its (stable) real-valued solution Θ with a straightforward iterative algorithm (Binder and Pesaran (1995, 1997)). Alternatively, the stable solution of the quadratic matrix equation in (5) can be found in closed-form from the solution to a related generalized eigenvalue problem.

I construct a pair of $2k \times 2k$ companion matrix forms (D, E) where the $k \times k$ matrices Φ_1 and Φ_2 that describe (5) enter as follows:

$$D = \begin{bmatrix} I_k & -\Phi_1 \\ I_k & 0_k \end{bmatrix}, \ E = \begin{bmatrix} \Phi_2 & 0_k \\ 0_k & I_k \end{bmatrix}. \tag{92}$$

The solution to the generalized eigenvalue problem for the matrix pair (D, E) is a set of $2k$ eigenvalues q_k and their corresponding eigenvectors v_k such that $Dv_k = Ev_k q_k$. Assuming there are at least k stable eigenvalues (those who are inside the unit circle), I can then order the eigenvalues and their corresponding eigenvectors so that the k stable eigenvalues come first. I denote the diagonal matrix with the ordered eigenvalues as Q and the matrix of corresponding eigenvectors as V.

Partitioning the ordered matrix of eigenvalues Q to collect the first k stable eigenvalues and the remaining ones into block matrices and partitioning the matrix of eigenvectors V accordingly, I can write $Q \equiv \begin{bmatrix} Q^1 & 0_k \\ 0_k & Q^2 \end{bmatrix}$ and $V \equiv \begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix}$ where each block matrix is of dimension $k \times k$. The generalized eigenvalue problem can then be stated in matrix form as:

$$\begin{bmatrix} I_k & -\Phi_1 \\ I_k & 0_k \end{bmatrix} \begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix} = \begin{bmatrix} \Phi_2 & 0_k \\ 0_k & I_k \end{bmatrix} \begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix} \begin{bmatrix} Q^1 & 0_k \\ 0_k & Q^2 \end{bmatrix}, \tag{93}$$

and from here it follows that

$$\begin{bmatrix} V^{11} - \Phi_1 V^{21} & V^{12} - \Phi_1 V^{22} \\ V^{11} & V^{12} \end{bmatrix} = \begin{bmatrix} \Phi_2 V^{11} & \Phi_2 V^{12} \\ V^{21} & V^{22} \end{bmatrix} \begin{bmatrix} Q^1 & 0_k \\ 0_k & Q^2 \end{bmatrix} = \begin{bmatrix} \Phi_2 V^{11} Q^1 & \Phi_2 V^{12} Q^2 \\ V^{21} Q^1 & V^{22} Q^2 \end{bmatrix}. \tag{94}$$

This block system implies that $V^{11} = V^{21} Q^1$ and $V^{11} - \Phi_1 V^{21} = \Phi_2 V^{11} Q^1$. Substituting the first expression into the second one gives that $V^{21} Q^1 - \Phi_1 V^{21} = \Phi_2 V^{21} Q^1 Q^1$. Then, post-multiplying both sides by $\left(V^{21} \right)^{-1}$ and re-arranging, it follows that:

$$\Phi_2 \left(V^{21} \left(Q^1 \right)^2 \left(V^{21} \right)^{-1} \right) - \left(V^{21} Q^1 \left(V^{21} \right)^{-1} \right) + \Phi_1 = 0_k. \tag{95}$$

Defining Θ to be $\Theta \equiv V^{21} Q^1 \left(V^{21} \right)^{-1}$ gives $\Theta^2 = V^{21} Q^1 \left(V^{21} \right)^{-1} V^{21} Q^1 \left(V^{21} \right)^{-1} = V^{21} \left(Q^1 \right)^2 \left(V^{21} \right)^{-1}$. Therefore, under this definition of Θ, the matrix equation in (95) is observationally equivalent to the quadratic matrix equation (5) introduced in Section 2.1 for the purpose of decoupling the backward-looking and forward-looking terms of the LRE model given by (1). Hence, this suffices to characterize a solution of the

quadratic matrix equation based on its k stable (real-valued) eigenvalues.

If such a real-valued solution of the quadratic matrix equation exists with all its eigenvalues inside the unit circle, then it is straightforward to compute by constructing the matrices D and E and finding the ordered solution to the corresponding generalized eigenvalue problem for the matrix pair (D, E). The block matrix Q^1 contains the k stable eigenvalues and the block matrix V^{21} contains the corresponding eigenvectors from the generalized eigenvalue problem after partitioning, then the solution Θ for the quadratic matrix is simply $\Theta \equiv V^{21} Q^1 \left(V^{21} \right)^{-1}$. To check that this transformation Θ is valid one only needs to verify that all the eigenvalues collected in Q^1 are indeed inside the unit circle.[23]

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[23] The accompanying codes provided with the paper include a Matlab function that implements the iterative solution of Binder and Pesaran (1995, 1995) together with another Matlab function that obtains the solution using alternatively the generalized eigenvalue problem algorithm described in this Appendix. For a discussion of the quadratic matrix equation and the iterative algorithm, the interested reader is also refered to Binder and Pesaran (1995, 1997) and their accompanying Matlab codes: https://ideas.repec.org/c/dge/qmrbcd/73.html.

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