Online Appendix to A Theory of Capital Flow Retrenchment

J. Scott Davis and Eric van Wincoop

Globalization Institute Working Paper 422 Appendix
August 2023

Research Department

https://doi.org/10.24149/gwp422app

Online Appendix

A Theory of Capital Flow Retrenchment

J. Scott Davis and Eric van Wincoop

August 2023

This Online Appendix has 2 sections. The first section presents various analytical proofs for results in Sections 2 and 3 of the paper. The second section presents some additional details related to the numerical model in Section 4 of the paper, including a more complete discussion of the calibration of the within-country heterogeneity parameters Γ and κ. Throughout this Online Appendix any equation references that are not preceded by a letter refer to equations in the main text of the paper. Equation references that are preceded by a letter refer to equations in this Online Appendix.

A Proofs from paper

A.1 Proof of Theorem 1

With period 1 dividends of 1, $R_0 = (1+a)/a$ and $Q_{n,0} = Q_{n,1} = a$, (7) implies that $R_1^{p,i,n} = (1+a)/a$. We have $W_{n,0}^i - C_{n,0}^i = \beta W_{n,0}^i = a/\bar{z}$, so that from (6) $W_{n,1}^i = (1+a)/\bar{z}$ for all investors. Substituting $\bar{D}$ from Assumption 1, as well as $Q_n = a$ and $R = (1+a)/a$, into the portfolio expressions (16)-(17) gives time 1 portfolio shares that are the same as the time zero portfolio shares (21)-(22). Substituting these portfolio expressions, as well as $W_n^i = (1+a)/\bar{z}$ and $Q_n = a$, into the risky asset market clearing conditions (19), the markets clear in period 1 under Assumption 1 about K_n. The aggregate asset market clearing condition (20) also holds in period 1, after substituting $B_1 = B_0$, $W_n^i = (1+a)/\bar{z}$, $Q_{n,1} = a$ and the expression for B_0 in Assumption 1.

Since $R_t = 1/\beta$ for all $t \geq 1$, first-order condition (3) implies that borrower consumption is constant over time. Since income is constant, this implies $C_t^b = Y - B_0/a$. The borrower budget constraint (2) then implies $B_t = B_0$ for all $t \geq 1$. Since there is no uncertainty starting in period 2, we must have $R_t = (Q_{n,t+1} + D_{n,2})/Q_{n,t}$ for $t \geq 2$. This is satisfied when $R_t = (1+a)/a$, $Q_{n,t} = Q_{n,t+1} = (a/(1+a))D_n = aD_{n,2}$. Investor wealth remains constant after period 2 since $W_{n,t+1}^i = \beta R_t W_{n,t}^i$ for $t \geq 2$ and $R_t = 1/\beta$.

We finally need to check the aggregate asset market clearing condition (20) for $t \geq 2$. Since borrower safe asset holdings, investor wealth and asset prices remain constant from period 2 onward, we only need to check it for $t = 2$. We have

$$\sum_{n=1}^{N+1} \int_0^1 W_{n,2}^i di = \beta \frac{1+a}{\bar{z}} \sum_{n=1}^{N+1} \int_0^1 R^{p,i,n} di = \frac{a}{\bar{z}} R(N+1) + \frac{a}{\bar{z}} \sum_{n=1}^{N+1} \sum_{m=1}^{N+1} \int_0^1 z_{n,m}^i di \frac{D_m - RQ_m}{Q_m}$$

From (21)-(22), $\sum_{n=1}^{N+1} \int_0^1 z_{n,m}^i di = \bar{z} K_m$. Therefore

$$\sum_{n=1}^{N+1} \int_0^1 W_{n,2}^i di = \frac{(1+a)(N+1)}{\bar{z}} + \sum_{m=1}^{N+1} K_m \left(D_m - (1+a)\right)$$

Using $B_2 = B_0$, the period 2 aggregate asset market equilibrium can then be written as

$$\frac{1}{\bar{z}} a(N+1) + \frac{a}{1+a} \sum_{n=1}^{N+1} D_n K_n - a \sum_{n=1}^{N+1} K_n = \sum_{n=1}^{N+1} Q_{n,2} K_n + (N+1) B_0$$

Using $Q_{n,2} = (a/(1+a)) D_n$ and the expression for B_0 in Assumption 1, it is immediate that this is satisfied.

We finally point out that the conjectured value functions are correct. We conjectured $V_{n,1}^i = \alpha_{1,i} W_{n,1}^i$ and $V_{n,t}^i = \alpha_2 W_{n,t}^i$ for $t \geq 2$. First substituting the latter into the Bellman equation (8) for $t \geq 2$, together with $C_{n,t}^i = (1-\beta) W_{n,t}^i$ and $W_{n,t+1}^i = W_{n,t}^i$, we have $\alpha_2 = 1 - \beta$. Substituting $V_{n,1}^i = \alpha_{1,i} W_{n,1}^i$ into the Bellman equation (8) at time 1, together with $C_n^i = (1-\beta) W_n^i$ and $W_{n,2}^i = \beta R^{p,i,n} W_n^i$, we have

$$ln(\alpha_{1,i}) = ln(1-\beta) + \frac{\beta}{1-\beta} ln(\beta) + \frac{\beta}{1-\beta} \frac{1}{1-\gamma_i} ln\left(E(R^{p,i,n})^{1-\gamma_i}\right)$$

Substituting the portfolio shares (21)-(22), $Q_m = a$ and $R = a/(1+a)$ into the portfolio return expression (15), $\alpha_{1,i}$ becomes a function of structural model parameters.

A.2 Asset Price Changes

Here we will derive equations (25) and (26), which show the derivatives of asset prices with respect to G at $G = 1$. Since all risky asset prices are identical, they are denoted Q. The risky asset price Q and interest rate R can be jointly solved from the asset market clearing conditions. After substituting optimal portfolio shares, investor wealth

$$W_n^i = \frac{1}{\bar{z}}\left(1 + a + z_n^i(Q - a)\right) \tag{A.1}$$

and the borrower budget constraint (2), the asset market clearing conditions become

$$\frac{G}{\sigma^2}\left(\frac{1+a}{\bar{z}} + \frac{E\psi_i^2}{\bar{\psi}^2}(Q - a)\right)\left(\bar{D} - RQ\right) = \frac{1+a}{a\bar{\psi}} \tag{A.2}$$

$$(1 - R_0)B_0 + Y - C_1^b - \frac{Q - a}{1 + a} = 0 \tag{A.3}$$

Equation (A.2) is the risky asset market equilibrium condition. The left hand side of (A.2) shows that demand for risky assets depends both positively and negatively on the risky asset price. On the one hand, a rise in Q lowers the expected return on risky assets, lowering its demand. On the other hand, it raises wealth, which raises demand for risky assets. We adopt a rather weak assumption to make sure that the first effect dominates:

$$\frac{\bar{\psi}^3}{E\psi^2}(1 + a)^2 > \bar{z}^2\sigma^2 \tag{A.4}$$

In the absence of within-country heterogeneity, it implies that in the pre-shock equilibrium $\frac{\bar{D}}{Q} - R < R_0^2/\bar{z}$. With $\bar{z} < 1$, this condition says that the expected excess return on risky assets must be less than a number that is above 1, or 100 percent. This is evidently a very weak condition.

Equation (A.3) is the aggregate asset market clearing condition. Using $C_2^b = Y - (R/(1 + a))B_1$, the first-order condition of borrowers and their budget constraint, we can derive the following period 1 consumption of borrowers:

$$C^b = \frac{1}{1 + a^\rho(1 + a)^{1-\rho}R^{\rho-1}}\left(Y + \frac{1+a}{R}Y - \frac{1+a}{a}B_0\right) \tag{A.5}$$

Taking the derivative of (A.5) at the pre-shock equilibrium, we have

$$\frac{\partial C_1^b}{\partial R} = -\left(Y + (\rho - 1)\,C^b\right)\frac{a^2}{(1 + a)^2} \equiv -\lambda \tag{A.6}$$

where $C^b = Y + 1 - (1/\bar{z})$ is constant borrower consumption in the pre-shock equilibrium. Assuming $\bar{z} < 1$, and of course $C^b > 0$, it is easily checked that $\lambda > 0$.

Taking derivatives of the market clearing conditions, the changes in Q and R in response to a change in G are:

$$\frac{dQ}{dG} = \frac{\bar{z}(1+a)\frac{\sigma^2}{\bar{\psi}}}{(1+a)^2 - \frac{E\psi^2}{\bar{\psi}^2}\frac{\bar{z}^2\sigma^2}{\bar{\psi}} + \frac{a^2}{\lambda}} \tag{A.7}$$

$$\frac{dR}{dG} = \frac{1}{\lambda(1+a)}\frac{dQ}{dG} \tag{A.8}$$

Here $E\psi^2$ is the mean of ψ_i^2 across investors. If assumption (A.4) holds, the denominator of (A.7) is clearly positive. Since $\lambda > 0$, it follows that both Q and R drop in response to a drop in G.

A.3 Proof of Theorem 3

Use that $z^i = \alpha^i \bar{z}$ with $\alpha^i = \psi^i/\bar{\psi}$. (23)-(24) can then be written as

$$OF_n^{risky} = IF_n^{risky} = \frac{a}{1+a}Q\bar{\psi}G\frac{\bar{D}-RQ}{\sigma^2}\int_0^1 z_F^i\alpha^i\left(\frac{1+a}{\bar{z}} + \alpha^i(Q-a)\right)di - QE(z_F\alpha) \tag{A.9}$$

with z and z_F defined in (27) and (31). Substituting (A.2), we have

$$OF_n^{risky} = IF_n^{risky} = Q\frac{\frac{1+a}{\bar{z}}E(z_F\alpha) + (Q-a)E(z_F\alpha^2)}{\frac{1+a}{\bar{z}} + (1+var(\alpha))(Q-a)} - QE(z_F\alpha) \tag{A.10}$$

This uses that $1 + var(\alpha) = E(\alpha^2) = E(\psi/\bar{\psi})^2 = (E\psi^2)/\bar{\psi}^2$. Differentiating with respect to Q at $Q = a$ gives

$$dOF_n^{risky} = dIF_n^{risky} = \frac{a}{1+a}\bar{z}\left(Ez_F\alpha(\alpha - 1 - var(\alpha))\right)dQ \tag{A.11}$$

It follows that both outflows and inflows of risky assets go down equally in response to an increase in global risk-aversion as long as $Ez_F\alpha(\alpha - 1 - var(\alpha)) > 0$. Using that the expectation of $\alpha(\alpha - 1 - var(\alpha))$ is equal to zero, this is the case when $cov(z_F, \alpha(\alpha - 1 - var(\alpha))) > 0$ or $cov(z_F, Z) > 0$, which is Assumption 2. This establishes that gross capital flows drop in response to the shock.

We finally show that $cov(z_F, Z) > 0$ is satisfied under a specific set of assumptions about the distributions of κ and Γ discussed in Section 3. Assume that $\kappa_i = \bar{\kappa} + \epsilon_i^\kappa$ and $\Gamma_i = \bar{\Gamma} + \omega\epsilon_i^\kappa + \epsilon_i^\Gamma$, where ϵ_i^κ and ϵ_i^Γ are uncorrelated, have symmetric distributions, and have mean zero. They are also such that Γ_i and κ_i are always positive. We then show that $cov(z_F, Z) > 0$ as long as $\omega \geq 0$ and $var(\epsilon^\kappa) > 0$.

This means that we must have cross-sectional variation in home bias. We do not necessarily need cross-sectional variation in risk-aversion.

Define $\eta = N\Gamma\kappa$. Using that $\alpha = \psi/E(\psi)$ and $z_F\alpha = \eta/E(\psi)$, we can write the condition $cov(z_F, Z) > 0$ as

$$cov(\eta,\psi)E(\psi) - var(\psi)E\eta > 0 \tag{A.12}$$

Using that $\kappa = \bar{\kappa} + \epsilon^{\kappa}$ and $\Gamma = \bar{\Gamma} + \omega\epsilon^{\kappa} + \epsilon^{\Gamma}$, we have

$$\psi = \bar{\Gamma}(1+N\bar{\kappa}) + (\omega + N\omega\bar{\kappa} + N\bar{\Gamma})\epsilon^{\kappa} + (1+N\bar{\kappa})\epsilon^{\Gamma} + N\epsilon^{\kappa}\epsilon^{\Gamma} + \omega N\left(\epsilon^{\kappa}\right)^2 \tag{A.13}$$
$$\eta = N\bar{\Gamma}\bar{\kappa} + N(\omega\bar{\kappa} + \bar{\Gamma})\epsilon^{\kappa} + N\bar{\kappa}\epsilon^{\Gamma} + N\epsilon^{\kappa}\epsilon^{\Gamma} + \omega N\left(\epsilon^{\kappa}\right)^2 \tag{A.14}$$

Using the assumed properties of ϵ^{κ} and ϵ^{Γ} (independent, symmetric distributions), it follows that

$$\begin{aligned}
var(\psi) &= (\omega + N\omega\bar{\kappa} + N\bar{\Gamma})^2 var(\epsilon^{\kappa}) + (1+N\bar{\kappa})^2 var(\epsilon^{\Gamma}) \\
&\quad + N^2 var(\epsilon^{\Gamma})var(\epsilon^{\kappa}) + \omega^2 N^2 E\left(\epsilon^{\kappa}\right)^4 - \omega^2 N^2\left[var(\epsilon^{\kappa})\right]^2 \\
cov(\eta,\psi) &= (\omega + N\omega\bar{\kappa} + N\bar{\Gamma})N(\omega\bar{\kappa} + \bar{\Gamma})var(\epsilon^{\kappa}) + (1+N\bar{\kappa})N\bar{\kappa}var(\epsilon^{\Gamma}) \\
&\quad + N^2 var(\epsilon^{\Gamma})var(\epsilon^{\kappa}) + \omega^2 N^2 E\left(\epsilon^{\kappa}\right)^4 - \omega^2 N^2\left[var(\epsilon^{\kappa})\right]^2 \\
E(\psi) &= \bar{\Gamma}(1+N\bar{\kappa}) + \omega N var(\epsilon^{\kappa}) \\
E(\eta) &= N\bar{\Gamma}\bar{\kappa} + \omega N var(\epsilon^{\kappa})
\end{aligned}$$

Then collecting terms, we have

$$\begin{aligned}
cov(\eta,\psi)E(\psi) - var(\psi)E\eta &= (\omega + N\omega\bar{\kappa} + N\bar{\Gamma})N\bar{\Gamma}^2 var(\epsilon^{\kappa}) + \bar{\Gamma}N^2 var(\epsilon^{\Gamma})var(\epsilon^{\kappa}) + \bar{\Gamma}\omega^2 N^2 E\left(\epsilon^{\kappa}\right)^4 \\
&\quad -\omega^2 N(\omega + N\omega\bar{\kappa} + 2N\bar{\Gamma})\left[var(\epsilon^{\kappa})\right]^2 - \omega N(1+N\bar{\kappa})var(\epsilon^{\Gamma})var(\epsilon^{\kappa})
\end{aligned}$$

Rewrite this as

$$\begin{aligned}
cov(\eta,\psi)E(\psi) - var(\psi)E\eta &= \bar{\Gamma}N^2 var(\epsilon^{\Gamma})var(\epsilon^{\kappa}) + \left(\bar{\Gamma}^2 - var(\epsilon^{\Gamma}) - \omega^2 var(\epsilon^{\kappa})\right)(1+N\bar{\kappa})N\omega var(\epsilon^{\kappa}) \\
&\quad \left(\omega^2 E\left(\epsilon^{\kappa}\right)^4 - 2\omega^2\left[var(\epsilon^{\kappa})\right]^2 + \bar{\Gamma}^2 var(\epsilon^{\kappa})\right)N^2\bar{\Gamma}
\end{aligned} \tag{A.15}$$

The first term of (A.15) is clearly greater than or equal to zero. Next consider the second term in the first line of (A.15). Since Γ is assumed to be positive, we have $\omega\epsilon^{\kappa} + \epsilon^{\Gamma} > -\bar{\Gamma}$. Since ϵ^{κ} and ϵ^{Γ} are symmetrically distributed and have a mean of zero, it follows that $(\omega\epsilon^{\kappa} + \epsilon^{\Gamma})^2 < \bar{\Gamma}^2$. Taking the expectation, we have

$$\omega^2 var(\epsilon^{\kappa}) + var(\epsilon^{\Gamma}) < \bar{\Gamma}^2 \tag{A.16}$$

This implies that the second term of the first line of (A.15) is positive since $var(\epsilon^\kappa) > 0$. Finally consider the last term of (A.15). We have

$$E\left(\epsilon^\kappa\right)^4 = var\left((\epsilon^\kappa)^2\right) + \left[var(\epsilon^\kappa)\right]^2 \geq \left[var(\epsilon^\kappa)\right]^2$$

Therefore the term in brackets in the last term of (A.15) is

$$\omega^2 E\left(\epsilon^\kappa\right)^4 - 2\omega^2\left[var(\epsilon^\kappa)\right]^2 + \bar{\Gamma}^2 var(\epsilon^\kappa) \geq \left(\bar{\Gamma}^2 - \omega^2 var(\epsilon^\kappa)\right) var(\epsilon^\kappa)$$

From (A.16) this is positive, which completes the proof that $cov(\eta, \psi)E(\psi) - var(\psi)E\eta > 0$ and therefore $cov(z_F, Z) > 0$.

A.4 Period 2 Equilibrium

The paper focuses on the period 1 equilibrium, taken as given that the period 2 risky asset prices are equal to $Q_{n,2} = (a/(1+a))D_n$. This affects returns of risky assets from period 1 to 2, which affects portfolios. We will show that this indeed holds, even after a shock to G.

We will show that $Q_{n,t} = (a/(1+a))D_n$ and $R_t = (1+a)/a$ is an equilibrium for all $t \geq 2$. We need to check two things. First, since there is no uncertainty from time 2 onwards, all assets need to have the same deterministic return. Second, the aggregate asset market clearing condition needs to hold from time 2 onwards. Regarding the first, the return on "risky" assets is

$$\frac{Q_{n,t+1} + D_{n,2}}{Q_{n,2}} \tag{A.17}$$

Here $D_{n,2}$ is the period 2 dividend that is constant from then on. Recall that $D_n = D_{n,2}/(1-\beta) = (1+a)D_{n,2}$. Substituting $Q_{n,t} = (a/(1+a))D_n$ for all $t \geq 2$ and $D_{n,2} = D_n/(1+a)$, (A.17) becomes $(1+a)/a$, which is R_t from time 2 onwards. So the first part checks out.

For the second part we need to check that

$$\frac{a}{1+a}\sum_{n=1}^{N+1}\int_0^1 W_{n,t}^i di = \sum_{n=1}^{N+1} Q_{n,t}K_n + (N+1)B_t \tag{A.18}$$

for $t \geq 2$.

First consider borrower debt. Assuming that our conjecture that the interest rate is $(1+a)/a$ from time 2 onward is correct, we have

$$B_2 = R_1 B_1 - Y + C_2^b \tag{A.19}$$

$$B_t = \frac{1+a}{a}B_{t-1} - Y + C_t^b \quad t \geq 3 \tag{A.20}$$

We also know from the first-order condition for consumption that consumption will be constant from time 2 onwards. The solution is

$$B_t = B_2 = \frac{a}{1+a} R_1 B_1 \tag{A.21}$$

$$C_t^b = Y - \frac{1}{a} B_2 \tag{A.22}$$

for $t \geq 2$.

Now return to (A.18). Start with $t = 2$. The term $\sum_{n=1}^{N+1} \int_0^1 W_{n,2}^i di$ is the value of the wealth of all investors at the start of period 2, before consumption. This includes all investors within and across countries. We know that in the aggregate they hold the following quantities of assets in period 1. They hold K_n of country n risky assets. Since the global supply of safe assets is zero, their safe asset holdings in period 1 are equal to total borrower debt, so $(N+1)B_1$. Each safe asset earns a return of R_1 in period 2. Each country n risky asset earns $Q_{n,2} + D_{n,2}$ in period 2. Therefore we have

$$\sum_{n=1}^{N+1} \int_0^1 W_{n,2}^i di = (N+1) B_1 R_1 + \sum_{n=1}^{N+1} K_n (Q_{n,2} + D_{n,2}) \tag{A.23}$$

Substitute this back into (A.18) for $t = 2$. Also substitute $B_2 = (a/(1+a))R_1 B_1$, $Q_{n,2} = (a/(1+a))D_n$ and $D_{n,2} = D_n/(1+a)$. This gives

$$(N+1)\frac{a}{1+a} R_1 B_1 + \frac{a}{1+a} \sum_{n=1}^{N+1} K_n D_n = \frac{a}{1+a}(N+1) B_1 R_1 + \sum_{n=1}^{N+1} K_n \frac{a}{1+a} D_n \tag{A.24}$$

This is clearly satisfied.

Finally consider (A.18) for $t \geq 3$. At time $t-1$ investors hold K_n risky assets of country n. Their holdings of safe assets is equal to borrower debt, so $(N+1)B_2$. Therefore we have

$$\sum_{n=1}^{N+1} \int_0^1 W_{n,t}^i di = (N+1) \frac{1+a}{a} B_2 + \sum_{n=1}^{N+1} K_n (Q_{n,t} + D_{n,2}) \tag{A.25}$$

for $t \geq 3$. Substitute this back into (A.18) for $t \geq 3$. Also substitute $B_t = B_2$, $Q_{n,t} = (a/(1+a))D_n$ and $D_{n,2} = D_n/(1+a)$. This gives

$$(N+1) B_2 + \frac{a}{1+a} \sum_{n=1}^{N+1} K_n D_n = (N+1) B_2 + \sum_{n=1}^{N+1} K_n \frac{a}{1+a} D_n \tag{A.26}$$

This is clearly satisfied.

B Additional details for the numerical model

In this section we present some additional detail for the model in the quantitative section of the paper. First we discuss adding cross-country correlation in dividends to the model to make it more realistic. Then we go into greater detail about the calibration of the key model parameters.

B.1 Cross-country correlation in dividends

For analytical tractability, until Section 4 the paper assumes that dividends are uncorrelated across countries. For the numerical exercise in Section 4 we relax this assumption in order to make the model and calibration more realistic.

Assume that

$$D_m = D + F_m \tag{B.1}$$

where D is a common component and F_m is an idiosyncratic component. The expectation of D_m is $\bar{D}$. Assume that D and F_m are uncorrelated and that F_m is uncorrelated across countries. Assume that for country n investors the variance of F_n is σ^2, while for investor i in country n the variance of F_m, with $m \neq n$, is σ^2/κ_i. Also let σ_d^2 be the variance of D. In what follows we will take the perspective of investor i in country 1. Once we derive an expression for the optimal portfolio of investor i in country 1, it is then straightforward to generalize this to investor i in any country n.

Let Σ^i be the covariance of the vector $[D_1, ..., D_{N+1}]'$ for investor i in country 1. It follows that

$$\Sigma^i = A^i + \sigma_d^2 \iota\iota' \tag{B.2}$$

where ι is a $(N+1)$ by 1 vector of ones and A^i is a diagonal matrix with $A^i_{1,1} = \sigma^2$ and the other diagonal elements equal to σ^2/κ_i. We have

$$[\Sigma^i]^{-1} = [A^i]^{-1} - \frac{\sigma_d^2}{1 + \sigma_d^2 \iota' [A^i]^{-1} \iota} [A^i]^{-1} \iota\iota' [A^i]^{-1} \tag{B.3}$$

$[A^i]^{-1}$ is a diagonal matrix with $1/\sigma^2$ in element (1,1) and κ_i/σ^2 in the other diagonal elements.

We have

$$[A^i]^{-1}\iota = \frac{1}{\sigma^2}\begin{pmatrix} 1 \\ \kappa_i \\ ... \\ \kappa_i \end{pmatrix} \tag{B.4}$$

and

$$\iota'[A^i]^{-1}\iota = \frac{1+N\kappa_i}{\sigma^2} \tag{B.5}$$

and

$$[A^i]^{-1}\iota\iota'[A^i]^{-1} = \frac{1}{\sigma^4}\begin{pmatrix} 1 & \kappa_i & ... & \kappa_i \\ \kappa_i & \kappa_i^2 & ... & \kappa_i^2 \\ ... & ... & ... & ... \\ \kappa_i & \kappa_i^2 & ... & \kappa_i^2 \end{pmatrix} \tag{B.6}$$

Define

$$\eta_i = \frac{\sigma_d^2}{\sigma^2 + \sigma_d^2(1+N\kappa_i)} \tag{B.7}$$

and define $\nu = \frac{\sigma_d^2}{\sigma_d^2+\sigma^2}$. This is the cross-country correlation of dividends. Then

$$\eta_i = \frac{\nu}{1-\nu+\nu(1+N\kappa_i)} \tag{B.8}$$

Then

$$[\Sigma^i]^{-1} = \frac{1}{\sigma^2}\begin{pmatrix} 1-\eta_i & -\eta_i\kappa_i & -\eta_i\kappa_i & ... & -\eta_i\kappa_i & -\eta_i\kappa_i \\ -\eta_i\kappa_i & \kappa_i-\eta_i\kappa_i^2 & -\eta_i\kappa_i^2 & ... & -\eta_i\kappa_i^2 & -\eta_i\kappa_i^2 \\ ... & ... & ... & ... & ... & ... \\ -\eta_i\kappa_i & -\eta_i\kappa_i^2 & -\eta_i\kappa_i^2 & ... & -\eta_i\kappa_i^2 & \kappa_i-\eta_i\kappa_i^2 \end{pmatrix} \tag{B.9}$$

Next consider the portfolio problem for agent i in country 1. The agent maximizes

$$E\left(R^{p,i,1}\right) - 0.5\gamma_{i,1} var\left(R^{p,i,1}\right) \tag{B.10}$$

where

$$R^{p,i,1} = R + \sum_{m=1}^{N+1} z_{1,m}^i\left(\frac{D_m - RQ_m}{Q_m}\right) \tag{B.11}$$

Define the portfolio vector of agent i in country 1 as

$$z_1^i = \begin{pmatrix} z_{1,1}^i \\ ... \\ z_{1,N+1}^i \end{pmatrix} \tag{B.12}$$

The vector of expected excess returns is

$$\mu = \begin{pmatrix} \frac{\bar{D}-RQ_1}{Q_1} \\ ... \\ \frac{\bar{D}-RQ_{N+1}}{Q_{N+1}} \end{pmatrix} \tag{B.13}$$

The variance of the vector of excess returns is $\tilde{Q}\Sigma^i\tilde{Q}$, where $\tilde{Q}$ is a diagonal matrix with $1/Q_m$ in element (m,m) of the diagonal.

Investor i from country 1 then maximizes

$$\mu' z_1^i - 0.5\gamma_{i,1}\left(z_1^i\right)'\tilde{Q}\Sigma^i\tilde{Q}\left(z_1^i\right) \tag{B.14}$$

The optimal portfolio is

$$z_1^i = \frac{1}{\gamma_{i,1}}\left(\tilde{Q}\Sigma^i\tilde{Q}\right)^{-1}\mu \tag{B.15}$$

We can also write this as

$$z_1^i = \frac{1}{\gamma_{i,1}}\tilde{Q}^{-1}[\Sigma^i]^{-1}\tilde{Q}^{-1}\mu \tag{B.16}$$

$\tilde{Q}^{-1}$ is a diagonal matrix with Q_m in element (m,m). We then have

$$\tilde{Q}^{-1}[\Sigma^i]^{-1}\tilde{Q}^{-1} = \tag{B.17}$$
$$\frac{1}{\sigma^2}\begin{pmatrix} (1-\eta_i)Q_1^2 & -\eta_i\kappa_i Q_1Q_2 & -\eta_i\kappa_i Q_1Q_3 & ... & -\eta_i\kappa_i Q_1Q_N & -\eta_i\kappa_i Q_1Q_{N+1} \\ -\eta_i\kappa_i Q_1Q_2 & \left(\kappa_i-\eta_i\kappa_i^2\right)Q_2^2 & -\eta_i\kappa_i^2 Q_2Q_3 & ... & -\eta_i\kappa_i^2 Q_2Q_N & -\eta_i\kappa_i^2 Q_2Q_{N+1} \\ ... & ... & ... & ... & ... & ... \\ -\eta_i\kappa_i Q_1Q_{N+1} & -\eta_i\kappa_i^2 Q_2Q_{N+1} & -\eta_i\kappa_i^2 Q_3Q_{N+1} & ... & -\eta_i\kappa_i^2 Q_NQ_{N+1} & \left(\kappa_i-\eta_i\kappa_i^2\right)Q_{N+1}^2 \end{pmatrix}$$

Using that

$$\gamma_{i,1} = \frac{1}{\Gamma_i G} \tag{B.18}$$

the portfolio expressions become

$$z_{1,1}^i = \frac{Q_1\Gamma_i G}{\sigma^2}\left((1-\eta_i)(\bar{D}-RQ_1) - \eta_i\kappa_i\sum_{m\neq 1}(\bar{D}-RQ_m)\right) \tag{B.19}$$

and for $m \neq 1$

$$z_{1,m}^i = \frac{Q_m\Gamma_i G}{\sigma^2}\left(-\eta_i\kappa_i(\bar{D}-RQ_1) + (\kappa_i-\eta_i\kappa_i^2)(\bar{D}-RQ_m) - \eta_i\kappa_i^2\sum_{k\neq 1,m}(\bar{D}-RQ_k)\right) \tag{B.20}$$

For investor i from country n, these portfolio expressions become

$$z^i_{n,n} = \frac{Q_n \Gamma_i G}{\sigma^2} \left((1-\eta_i)(\bar{D} - RQ_n) - \eta_i \kappa_i \sum_{m \neq n} (\bar{D} - RQ_m) \right) \tag{B.21}$$

and for $m \neq n$

$$z^i_{n,m} = \frac{Q_m \Gamma_i G}{\sigma^2} \left(-\eta_i \kappa_i (\bar{D} - RQ_n) + (\kappa_i - \eta_i \kappa_i^2)(\bar{D} - RQ_m) - \eta_i \kappa_i^2 \sum_{k \neq n,m} (\bar{D} - RQ_k) \right) \tag{B.22}$$

Notice that the portfolio expressions (17) and (18) in the paper correspond to $\sigma^2_d = 0$, so that $\eta_i = 0$.

Next consider the portfolio expressions in the pre-shock equilibrium, which generalize (24) and (25) in the paper. In the pre-shock equilibrium we still have $Q_n = a$ for all n and $R = (1+a)/a$. We also have

$$\bar{D} = 1 + a + \frac{\sigma^2 \bar{z}}{a\bar{\psi}} \tag{B.23}$$

This remains the same as in equation (11) of the paper. The definition of ψ_i, and therefore $\bar{\psi}$, has changed. Define

$$\psi_i = \Gamma_i (1 + N\kappa_i)(1 - \eta_i(1 + N\kappa_i)) \tag{B.24}$$

with $\bar{\psi}$ defined as the mean across i of ψ_i. In the case where returns are uncorrelated, so that $\eta_i = 0$, this corresponds exactly to the ψ_i in the paper.

The portfolios in the pre-shock equilibrium are then

$$z^i_{n,n} = \bar{z}\frac{\Gamma_i(1 - \eta_i(1 + \kappa_i N)}{\bar{\psi}} \tag{B.25}$$

$$z^i_{n,m} = \bar{z}\frac{\Gamma_i \kappa_i (1 - \eta_i(1 + \kappa_i N)}{\bar{\psi}} \tag{B.26}$$

The mean portfolio share of investor i in all countries is

$$z^i = \bar{z}\frac{\Gamma_i(1 + N\kappa_i)(1 - \eta_i(1 + \kappa_i N)}{\bar{\psi}} = \bar{z}\frac{\psi_i}{\bar{\psi}} \tag{B.27}$$

Next we need to check that market clearing conditions are satisfied in period 1. If we start with $W^i_{n,0} = (1+a)/\bar{z}$ in period 0, it is immediate that in period 1 we still have $W_n^i = (1+a)/\bar{z}$ in the pre-shock equilibrium as the portfolio return is $(1+a)/a$. The risky asset market clearing conditions then imply

$$K_n = E(\Gamma(1+N\kappa)(1-\eta(1+\kappa N))\frac{1}{\bar{\psi}} \tag{B.28}$$

which is equal to 1.

It is easy to check that from the aggregate asset market clearing condition, it remains the case that in the pre-shock equilibrium

$$B_0 = a\left(\frac{1}{\bar{z}} - \frac{\sum_{n=1}^{N+1} K_n}{N+1}\right) \tag{B.29}$$

B.2 Calibration

In this section we discuss the calibration of the equity return correlation ν and the within-country heterogeneity parameters Γ_i and κ_i. For this we rely on the three Calvet et al. papers cited in the text that use the Swedish administrative data to discuss the within country heterogeneity of wealth and portfolio shares. The Calvet et al. (2009a) paper provides the motivation for the distribution of risky shares z_i. The Calvet et al. (2007) paper provides the motivation for the distribution of foreign shares z_i^F. The Calvet et al. (2009b) paper provides the correlation between these risky and foreign shares. With a distribution of risky and foreign shares z_i and z_i^F, it is then simple to back out distributions of Γ_i and κ_i.

We need to find the Γ_i and κ_i for I investors where I is a large number (we use 100,000). We start by generating two random $N(0,1)$ series, each with I elements, and a correlation of c. The value of c will be discussed below. Refer to these series as $\bar{x}_i$ and $\bar{y}_i$. We then convert these to two random $U(0,1)$ series $x_i = \Psi(\bar{x}_i)$ and $y_i = \Psi(\bar{y}_i)$, where $\Psi(.)$ is the cdf of the standard normal distribution. We set the risky asset share z_i equal to x_i. We set the relative Sharpe ratio loss, $RSRL_i$ for investor i, equal to y_i. This is a measure of portfolio diversification that we will discuss shortly.

Figure I of Calvet et al. (2009a) plots a histogram with the distribution of the risky shares across households in the Swedish administrative data. This data is for 1999-2002, a period of rapid risky asset price appreciation followed by a fall in risky asset prices. The distribution is centered around 0.5. It is left-skewed during a boom in risky asset prices, with a large mass of households holding a risky share greater that 0.5, and right-skewed during a bust, with a large mass of households holding a risky share less that 0.5. But on average the risky shares are close to uniformly distributed uniformly between 0 and 1. Therefore we set $z_i = x_i$, which has a $U(0,1)$ distribution.

The Swedish data do not provide direct information on the share of risky assets invested abroad, z_i^F. To extract information about z_i^F, we use data on the Sharpe ratios for individual households from Calvet et al. (2007). We first discuss how these Sharpe ratios are computed in the model.

The excess return for the portfolio of investor i in country n is:

$$R^{p,i,n} - R = \sum_{m=1}^{N+1} z_{n,m}^i \left(\frac{D_m - RQ_m}{Q_m} \right) \tag{B.30}$$

We use the portfolio shares from the pre-shock equilibrium, as well as the pre-shock asset prices $Q_m = a$ and $R = (1+a)/a$. (7) then becomes

$$\begin{aligned} R^{p,i,n} - R &= \Gamma_i \left(1 - \eta_i \left(1 + N\kappa_i\right)\right) \frac{1}{\bar{\psi}} \bar{z} \left(\frac{D_n - 1 - a}{a} \right) \\ &+ \sum_{m \neq n} \Gamma_i \kappa_i \left(1 - \eta_i \left(1 + N\kappa_i\right)\right) \frac{1}{\bar{\psi}} \bar{z} \left(\frac{D_m - 1 - a}{a} \right) \end{aligned} \tag{B.31}$$

The Sharpe ratio is equal to the expected excess return divided by the standard deviation of the excess return:

$$S_i = \frac{E\left(R^{p,i,n} - R\right)}{\left(var\left(R^{p,i,n} - R\right)\right)^{0.5}} \tag{B.32}$$

We have $E\left(R^{p,i,n} - R\right) = z_i \left(\bar{z} \frac{\sigma^2}{a^2 \bar{\psi}} \right)$ and

$$\begin{aligned} var\left(R^{p,i,n} - R\right) &= Var \begin{pmatrix} \Gamma_i \left(1 - \eta_i \left(1 + N\kappa_i\right)\right) \frac{1}{\bar{\psi}} \bar{z} \left(\frac{D_n - 1 - a}{a} \right) \\ + \sum_{m \neq n} \Gamma_i \kappa_i \left(1 - \eta_i \left(1 + N\kappa_i\right)\right) \frac{1}{\bar{\psi}} \bar{z} \left(\frac{D_m - 1 - a}{a} \right) \end{pmatrix} \\ &= z_i^2 Var \left(\left(1 - z_i^F\right) \left(\frac{D_n - 1 - a}{a} \right) + \sum_{m \neq n} \frac{z_i^F}{N} \left(\frac{D_m - 1 - a}{a} \right) \right) \\ &= z_i^2 \begin{pmatrix} \left(\left(1 - z_i^F\right)^2 + N \left(\frac{z_i^F}{N} \right)^2 \right) \left(\frac{\sigma^2 + \sigma_d^2}{a^2} \right) \\ + \left((N)(N-1) \frac{z_i^F}{N} + 2N \left(1 - z_i^F\right) \right) \frac{z_i^F}{N} \left(\frac{\sigma_d^2}{a^2} \right) \end{pmatrix} \end{aligned}$$

From Calvet et al. (2007), the relative Sharpe ratio loss of the portfolio of investor i is:

$$RSRL_i = 1 - \frac{S_i}{S_D} \tag{B.33}$$

where S_D is the Sharpe ratio of the portfolio with the internationally diversified portfolio (in our model $z_i^F = N/(1+N)$). After some simplification:

$$\frac{\left(\frac{1}{1+N}\right)(1+N\nu)}{\left(1-z_i^F\right)^2 + N\left(\frac{z_i^F}{N}\right) + \left(2N-(1+N)z_i^F\right)\frac{z_i^F}{N}\nu} = \left(1-RSRL_i\right)^2 \tag{B.34}$$

where $\nu = \frac{\sigma_d^2}{\sigma_d^2+\sigma^2}$ is the cross-country correlation of dividends. We can back out the value of this parameter using the data in Calvet et al. (2007). They report that the Sharpe ratio for the currency-hedged international benchmark portfolio is 45.2. Furthermore they report that the Sharpe ratio for the benchmark Swedish portfolio is 27.4. Holding the benchmark Swedish portfolio then implies a Sharpe ratio loss of $RSRL = 1 - \frac{27.4}{45.2}$. In the model, holding only Swedish risky assets implies $z_i^F = 0$. It then follows from (B.34) that $\nu = \frac{(N+1)(1-RSRL)^2-1}{N} = 0.33$.

Table 4 of Calvet et al. (2007) presents the cumulative distribution of the Sharpe ratio loss of individual portfolios relative to the international benchmark. We assume that the cdf of the relative Sharpe ratio loss (RSRL) across the I investors in our model matches the cdf of the RSRL from this table. The table presents the 25th, 50th, 75th, 90th, 95th, and 99th percentiles of the RSRL. Define $\Psi_{RSRL}(.)$ as the cdf of the RSRL. The table implies $\Psi_{RSRL}(0.89) = 0.99$, $\Psi_{RSRL}(0.69) = 0.95$, $\Psi_{RSRL}(0.55) = 0.9$, $\Psi_{RSRL}(0.42) = 0.75$, $\Psi_{RSRL}(0.35) = 0.5$ and $\Psi_{RSRL}(0.29) = 0.25$. We assume that the investor with the lowest Sharpe ratio loss has a RSRL of 0, so $\Psi_{RSRL}(0) = 0$, and the investor with the highest Sharpe ratio loss has a RSRL of 1, so $\Psi_{RSRL}(1) = 1$. We then assume that the RSRL cdf is piecewise linear between these values.

We next use the values of the series y_i, which has a $U(0,1)$ distribution, to create a series $RSRL_i$ consistent with the cdf for $RSRL$: $RSRL_i = \left(\Psi_{RSRL}\right)^{-1}(y_i)$. We can then use (B.34) to back out the series of z_i^F across our I investors. For a large set of investors this z_i^F will be negative. Recall that RSRL of the Swedish portfolio is around 0.4. Then under this piecewise mapping, about 35 percent of investors have a greater RSRL than the benchmark Swedish portfolio, the portfolio they would have in this model with $z_i^F = 0$. Since we are not concerned with domestic investment mistakes and only the lack of international diversification, we simply assume that these investors with a RSRL lower than the Swedish benchmark simply have a $z_i^F = 0$.[1]

[1] Calvet et al. (2007) also report the RSRL of investors relative to the Swedish benchmark

Once we know both the risky portfolio share z_i as well as z_i^F for our I investors, we can back out Γ_i and κ_i. We can obtain κ_i from $z_i^F = N\kappa_i/(1+N\kappa_i)$. Recall that the risky share is $z_i = \bar{z}\frac{\psi_i}{\bar{\psi}}$ where $\psi_i = \Gamma_i(1+N\kappa_i)(1-\eta_i(1+N\kappa_i))$, $\bar{\psi}$ is the mean of ψ_i, and $\eta_i = \frac{\nu}{1-\nu+\nu(1+N\kappa_i)}$. We can use this to back out $\Gamma_i/\bar{\Gamma}$. As discussed in the text, we assume that Γ_i has a mean $\bar{\Gamma}$ of 0.1.

Finally, we need to discuss the correlation c that we assume between the two random $N(0,1)$ series $\bar{x}_i$ and $\bar{y}_i$ we generate to construct the x_i and y_i series. This correlation c affects the correlation between z_i and $RSLR_i$. The latter correlation is important as it determines the correlation between the the risky asset share and the foreign share. Calvet et al. (2009b) report that the correlation between the risky share and the RSRL in the Swedish data is -0.49. That is, investors with a higher risky asset share tend to have a more diversified portfolio. Calvet et al. (2007) report that investor sophistication has a positive effect on both the risky portfolio share and portfolio diversification. We can set the correlation c in order to target a correlation of -0.49 between z_i and $RSLR_i$. This will be the case when $c = -0.51$. The correlation between the original random $N(0,1)$ series that we generate is therefore very close to the correlation between z_i and $RSLR_i$ that is created from these two series.

All of this results in a correlation between the risky share and the foreign share of about 0.43. The average risky share is 0.5 and the average foreign share is 0.24.

portfolio. They report that the 50th percentile investor has a RSRL relative to the domestic benchmark of -0.08 (meaning they have a higher Sharpe ratio than the domestic portfolio) and the 75th percentile investor has a RSRL relative to the domestic benchmark of 0.04. Extrapolating between these two, we would conclude that the 65th percentile investor has a relative Sharp ratio loss relative to the Swedish portfolio of 0, further evidence that investors in the 65th to 100th percentile have no international diversification